SEMI-T-ABSO FUZZY SUBMODULES AND SEMI-T-ABSO FUZZY MODULES

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Abstract. Let Ḿ be a unitary R-module and R be a commutative ring with identity and let X be a fuzzy module of an R-module Ḿ. Our aim in this paper to study the concepts semi T-ABSO fuzzy submodules and semi T-ABSO fuzzy modulesas generalizations of T-ABSO fuzzy submodules and T-ABSO fuzzy modules. Many new basic properties, characterizations and relationships between semi T-ABSO fuzzy submodules(modules) and other concepts are given.

Keywords. T-ABSO fuzzy submodule,T-ABSO fuzzy module, semi T-ABSO fuzzy ideal, semi T-ABSO fuzzy submodule, semi T-ABSO fuzzy module, quasi-prime fuzzy submodule, semiprime fuzzy submodule.

1. Introduction

Zahedi [17], in1992 presented the concept of a fuzzy ideal A fuzzy subset *K* of a ring R is called a fuzzy ideal of R, if $\forall\, x, y \in$ R: $K(x\text{-}y) \geq \min\{K(x), K(y)\}$ and $K(xy) \geq \max\{K(x),K(y)\}$".
Mukhrjee [13], in 1989 intoduced the concept of prime fuzzy ideal " A fuzzy ideal Ĥ of a ring R is called a prime fuzzy ideal if Ĥ is a non-empty and for all a_s, b_l fuzzy singletons of R such that $a_s b_l \subseteq$ Ĥ implies that either $a_s \subseteq$ Ĥ or $b_l \subseteq$ Ĥ, $\forall s,l \in [0,1]$".Deniz et al [3], in 2017 presented the concept of 2-absorbing fuzzy ideal which is a generalization of prime fuzzy ideal. Darani and Soheilnia [2], in 2011 introduced the concept of 2-absorbing submodule "a proper submodule N of Ḿ is called 2-absorbing submodule of Ḿ if whenever a , b ∈ R , m∈ Ḿ and abm ∈N , then am ∈N or bm ∈N or ab $\in(N:_R Ḿ)$". Hatam and wafaa [7], in 2018 expanded this concept " Let X be fuzzy module of an R-module Ḿ. A proper fuzzy submodule A of X is called T-ABSO fuzzy submodule if whenever a_s, b_l be fuzzy singletons of R, and $x_v \subseteq X$, $\forall\, s, l, v \in [0,1]$ such that $a_s b_l x_v \subseteq A$ then either $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$" Abdulrahman [1], in 2015 presented the definition of 2-absorbing module" An R-module Ḿ is called a 2-absorbing module if zero (0) submodule of Ḿ is 2-absorbing submodule " equivalently " if whenever a, b ∈ R , m∈ Ḿ and abm =0 , then am =0 or bm =0 or ab ∈ annḾ". Hadi [4], in 2004 presented the concept of semiprime fuzzy submodules "Let A be a fuzzy submodule of a fuzzy module X of an R-module Ḿ such that A≠ X, A is called semiprime fuzzy submodule if for each fuzzy singletone r_k of R, $x_v \subseteq X$, $r_k^2 x_v \subseteq A$ implies $r_k x_v \subseteq A$ " Maysoun [11], in 2012 introduced the concept of semiprime fuzzy module "Let X be a fuzzy module of an R-module Ḿ, X is called semiprime fuzzy module if for each non-empty fuzzy submodule A of X, F-annA is a semiprime fuzzy ideal of R".Hatam [6], in 2001 introduced the concept of quasi-prime fuzzy submodule" A fuzzy submodule *A* of a fuzzy module *X* of an R-module *M* is called a quasi-prime fuzzy submodule of *X* if whenever $a_s b_l x_v \subseteq A$ for fuzzy singletons a_s, b_l of R and $x_v \subseteq X$, $\forall s,l,v \in$ L, implies that $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$".Also Abdulrahman [1], in 2015 is circulated the concepts of 2-absorbing submodules and 2-absorbing modules to semi-2-absorbing submodules and semi-2-absorbing modules.

This paper be composed of two sections

In section (1) we present and study the concept of semi T-ABSO fuzzy submodule as a generalization of T-ABSO fuzzy submodule and we give many properties, characterizations and relationships between semi T-ABSO fuzzy and other concepts.
Futhermore we debate the direct sum of semi T-ABSO fuzzy submodules. In section(2) we present the concept of semi T-ABSO fuzzy modules , so many properties and characterizations are given . Also we debate the direct sum of semi T-ABSO fuzzy modules.
Note that we denote to fuzzy: F., module: M., submodule: subm. , [0,1]: L , otheroiwse: o.w.

2. Semi T-ABSO F. Subm.

In this section we present the concepts of semi-T-ABSO F. ideal and semi T-ABSO F. subm. Also introduced and study some properties and relations of semi-T F. subm. with other concepts of F. subm.

Frist we give the proposition specificates of T-ABSO F. subm. in terms of its level subm. is given:

"**Proposition 2.1.** Let A be T-ABSO F. subm. of F. M. X of an R- M. Ḿ iff the level subm. A_v is T-ABSO subm. of X_v, for all $v \in$ L, [7]".

Now, we present the concepts of a semi T-ABSO F. ideal.and semi T-ABSO F. subm. as follows:

Definition 2.2. A proper F. ideal Ĥ of a ring R is called a semi T-ABSO F. ideal if for F. singletons a_s, b_l of R such that $a_s^2 b_l \subseteq$ Ĥ ,

$\forall$ $s, l \in$ L, implies either $a_s b_l \subseteq$ Ĥ or $a_s^2 \subseteq$ Ĥ ; that is Ĥ a semi T-ABSO F. subm. of X of an R- M. R.

Definition 2.3. A proper F. subm. A of F. M. X of an R- M. Ḿ is called a semi T-ABSO F. subm. of X if for F. singletons a_s of R and $x_v \subseteq X$ such that $a_s^2 x_v \subseteq A$, $\forall s,v \in$ L, implies either $a_s x_v \subseteq A$ or $a_s^2 \subseteq (A:_R X)$.

The proposition specificates a semi T-ABSO F. subm. in terms of its level subm is given:

Proposition 2.4. Let A be F. subm. of F. M. X of an R- M. Ḿ. Then A is a semi T-ABSO F. subm. of X iff the level A_v is a semi T-ABSO subm. of X_v , $\forall$ $v \in$ L.

Proof. ($\Longrightarrow$) Let $a^2 x \in A_v$ for each a$\in$ R , $x \in X_v$, $\forall$ $v\in$ L, then $A(a^2 x) \geq v$, hence $(a^2 x)_v \subseteq A$ so that $a_s^2 x_k \subseteq A$ where v=min{s, k} and $(a^2)_s = a_s^2$. But A is a semi-T-ABSO F. subm., then either $a_s x_k \subseteq A$ or $a_s^2 \subseteq (A:_R X)$, hence $(ax)_v \subseteq A$ or $(a^2)_v \subseteq (A:_R X)$, implies $ax \in A_v$ or $a^2 \in (A_v:_R X_v)$. Thus A_v is a semi-T-ABSO of X_v.

($\Longleftarrow$) Let $a_s^2 x_k \subseteq A$ for F. singleton a_s of R and $x_v \subseteq X$, $\forall$ $s,k \in$ L, then $(a^2 x)_v \subseteq A$ where v=min{s, k}, hence A(a^2x)$\geq v$ so that $a^2 x \in A_v$. But A_v is a semi T-ABSO subm. of X_v, then either $ax \in A_v$ or $a^2 \in (A_v:_R X_v)$, hence $(ax)_v \subseteq A$ or $(a^2)_v \subseteq (A:_R X)$, so that $a_s x_k \subseteq A$ or $a_s^2 \subseteq (A:_R X)$. Thus A is a semi T-ABSO F. subm. of X.

Remarks and Examples 2.5

(1) Every semiprime F. subm. is a semi T-ABSO F. subm.

Proof:

Let $a_s^2 x_v \subseteq A$ for F. singleton a_s of R and $x_v \subseteq X$. Since semiprime F. subm., then that $a_s x_v \subseteq A$. So that A is a semi T-ABSO F. subm.

However the converse incorrect, for example:

Let X:Z→L such that X(y)= $\begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z- M. Z.

Let A: Z→L such that A(y) =$\begin{cases} \frac{1}{2} & if\ y \in 4Z \\ 0 & o.w. \end{cases}$

It is obvious that A is a fuzzy submodule of X.

Now, A is a semi T-ABSO fuzzy submodule of X since $2^2_{\frac{1}{3}}.1_{\frac{1}{3}}=4_{\frac{1}{3}} \subseteq A$, $2^2_{\frac{1}{3}} = 4_{\frac{1}{3}} \subseteq A$ where $A(4)=\frac{1}{2} > \frac{1}{3}$, but A is not semiprime fuzzy submodule since $2_{\frac{1}{3}}.\ 1_{\frac{1}{3}}=2_{\frac{1}{3}} \nsubseteq A$ because $A(2)=0 \ngtr \frac{1}{3}$.

(2) It obvious that every T-ABSO F. subm. is semi T-ABSO F. subm. However the convrse incorrect for example:

Let X:Z⊕Z→L such that X(x,y)= $\begin{cases} 1 & if\ (x,y) \in Z\oplus Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z- M. Z⊕Z.

Let A: Z⊕Z →L such that

A(x,y) =$\begin{cases} v & if\ (x,y) \in 10Z\oplus(0) \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A is F. subm. of X .

Now, A_v=10Z⊕(0) is not T-ABSO subm. in X_v=Z⊕Z as Z-M. since 2.5(1,0)=(10,0)∈10Z⊕(0), but 2(1,0)∉10Z⊕(0), 5(1,0)∉10Z⊕(0) and 2.5∉(10Z⊕(0):$_Z$ Z⊕Z)=(0). But A_v is a semi T-ABSO subm. since if r^2(x,0)∈A_v then r^2x∈10Z , hence it obvious that 10Z is semiprime, that is r x∈10Z, Thus r(x,0)∈10Z⊕(0)= A_v. Then A_v is a semi T-ABSO subm. Thus A is a semi T-ABSO F. subm.

(3) Every a quasi-prime F. subm. is a semi T-ABSO F. subm. However the converse incorrect. Consider the example in part(1) where A is semi T- ABSO F. subm., but A is not quasi-prime F. since $2_{\frac{1}{3}}.\ 2_{\frac{1}{3}}.\ 1_{\frac{1}{3}}=4_{\frac{1}{3}} \subseteq A$, but $2_{\frac{1}{3}}.\ 1_{\frac{1}{3}}=2_{\frac{1}{3}} \nsubseteq A$.

(4) Let A , B be F. subm. of F. M. X of an R- M. $\acute{M}$ and A⊆B. If A is a semi T-ABSO F. subm. of X then A is a semi T-ABSO F. subm. of B.

Proof. Let be F. singleton r_k of R and $x_v \subseteq B$ such that $r_k^2 x_v \subseteq A$, $\forall k,v \in$ L. Since B is F. subm. of X then $x_v \subseteq X$ and $r_k^2 x_v \subseteq A$, then either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. If $r_k^2 \subseteq (A:_R X)$ then $r_k^2 X \subseteq A$ and since B is F. subm. of X, hence $r_k^2 B \subseteq r_k^2 X$, so that $r_k^2 B \subseteq A$ implies $r_k^2 \subseteq (A:_R B)$. Thus A is a semi T-ABSO F. subm of B.

(5) The intersection of two semi T-ABSO F. subms is not necessary that a semi T- ABSO F. subm., for example:

Let X: Z_{12}→L such that X(y)= $\begin{cases} 1 & if\ y \in Z_{12} \\ 0 & o.w. \end{cases}$

It is clear that X is F. M. of Z- M. Z.

Let A: Z_{12}→L such that A(y) =$\begin{cases} v & if\ y \in \overline{(4)} \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

Let B: Z_{12}→L such that B(y) =$\begin{cases} v & if\ y \in \overline{(6)} \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A and B are F. subms of X .

Now, A_v=$\overline{(4)}$, B_v=$\overline{(6)}$ and X_v=Z_{12} as Z- M. It is obvious that A_v and B_v are semi T-ABSO subms, but $A_v \cap B_v$ =$\overline{(4)} \cap \overline{(6)}$ =$\overline{(0)}$ is not semi T-ABSO subm. since 2^2. $\overline{(3)}$= $\overline{(0)}$,but 2. $\overline{(3)} \neq \overline{(0)}$and $2^2 \notin annZ_{12} = 12Z$. So that A and B are semi T-ABSO F. subms, but A∩B is not a semi T-ABSO F. subm. of X.

(6) Let X:Z→L such that X(y)= $\begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X be F. M. of Z- M. Z.

Let A: Z→L such that A(y) =$\begin{cases} v & if\ y \in p^2 Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

Where p is a prime number.

It is obvious that A is F. subm. of X.

Now, $A_v = p^2 Z$ and $X_v = Z$ as Z- M

It is obvious that A_v , p is prime number is a semi T-ABSO subm. Thus A is a semi T-ABSO F. subm. of X.

(7) Let A, B be two F. subm. of F. M. X of an R- M. $\acute{M}$ suth that A≅B. If A is a semi T- ABSO F. subm. then it is not necessary that B is a semi T-ABSO F. subm.for example

Let X:Z→L such that X(y)= $\begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z- M. Z.

Let A: Z→L such that A(y) =$\begin{cases} v & if\ y \in 4Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

Let B: Z→L such that B(y) =$\begin{cases} v & if\ y \in 60Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A and B are F. subm. of X.

Now, A_v=4Z , B_v=60Z are subm.of X_v=Z as Z- M. and 4Z≅60Z , but $A_v = 4Z$ is semi T-ABSO while B_v=60Z is not semi T-ABSO subm. of X_v. So that A≅B where A is a semi T-ABSO F. subm., but B is not semi T-ABSO F. subm of X.

(8) If A is semi T-ABSO F. subm. of F. M. X of an R- M. $\acute{M}$ and $B \subseteq A$, it may be that B is not semi T-ABSO F. subm. for example:
Consider the example in part(7), where A is a semi T-ABSO F. subm., $B \subset A$ since $B_v = 60Z \subset A_v = 4Z$, but B is not semi T-ABSO F. subm. of X.

Recall that "Let A be a F. subm. of F. M. X of an R-module $\acute{M}$, then A is called an irreducible F. subm. if for all two F. subms B and K such that $B \cap K = A$ then $B = A$ or $K = A$ otherwise A is called reducible, [12]".

Proposition 2.6. Let X be F. M. of an R- M. Ḿ and A is irreducible F. subm. of X. Then the following expressions are equivalent:
1- A is T-ABSO F. subm. and $(A:_R X)$ is semi prime F. ideal.
2- A is a prime F. subm.
3- A is a semi prime F. subm.
4- A is a quasi prime F. subm.
5- A is T-ABSO F. subm. and $(A:_R X)$ is a prime F. ideal.
Proof. (1)⟹(2) Let $r_k(r_k x_v) \subseteq A$ for F. singleton r_k of R and $x_v \subseteq X$. Since A is T-ABSO F. subm., then $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. If $r_k x_v \subseteq A$ then we are done. If $r_k^2 \subseteq (A:_R X)$, then $r_k \subseteq (A:_R X)$ since $(A:_R X)$ is a semiprime F. ideal. so that A is a prime F. subm.
(1)⟹(3) Let $r_k^2 x_v \subseteq A$ for F. singletons r_k of R and $x_v \subseteq X$. Since A is T-ABSO F. subm., then $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. If $r_k x_v \subseteq A$ the proof is complete.
If $r_k^2 \subseteq (A:_R X)$, then $r_k \subseteq (A:_R X)$ since $(A:_R X)$ is a semi prime F. ideal. Hence $r_k x_v \subseteq A$. Thus A is a semi prime F. subm.
(2)⟹ (3)By [12].
(3)⟹ (4) By [6].
(4)⟹(5) Since A is a quasi prime F. subm., then A is T-ABSO F. subm. and $(A:_R X)$ is a prime F. ideal by [6].
(5)⟹(1) It is clear.

Proposition 2.7. Let X be F. M. of an R- M. $\acute{M}$ and A and B be F. subm. of X. Then A is a semi T-ABSO F. subm. iff $r_k^2 B \subseteq A$ for F. singleton r_k of R, $\forall k \in$ L, implies $r_k B \subseteq A$ or $r_k^2 \subseteq (A:_R X)$.
Proof. (⟹) Let $r_k^2 B \subseteq A$ for F. singleton r_k of R. Assume there exists $x_v \subseteq B$ such that $r_k x_v \nsubseteq A$, since $r_k^2 B \subseteq A$, hence $r_k^2 x_v \subseteq A$, but A is a semi T-ABSO F. subm. and $r_k x_v \nsubseteq A$.
Then $r_k^2 \subseteq (A:_R X)$.
(⟸) It is obvious.

Proposition 2.8. Let A be a proper F. subm. of F. M. of an R- M. Ḿ . If A is a semi T-ABSO F. subm. of X, then $(A:_R X)$ is a semi T-ABSO F. ideal.
Proof. Let a_s, b_l be F. singletons of R, such that $a_s^2 b_l \subseteq (A:_R X)$, hence $a_s^2 b_l X \subseteq A$, then $a_s^2 b_l x_v \subseteq A$ for each F. singleton $x_v \subseteq X$ and suppose that $a_s^2 \nsubseteq (A:_R X)$. Since A is a semi T-ABSO F. subm., hence $a_s b_l x_v \subseteq A$. So that $a_s b_l \subseteq (A:_R X)$. Then $(A:_R X)$ is semi T-ABSO F. ideal.

Recall that " A fuzzy module X of an R-module M is called a multiplication fuzzy module if for each non-empty fuzzy submodule A of X there exists a fuzzy ideal Ĥ of R such that A=ĤX,[6]".

The converse of Proposition (2.8) hold under the class of multiplication F. M. as follows:

Proposition 2.9. Let A be a proper F. subm. of a multiplication F. M. X of an R- M. Ḿ. If $(A:_R X)$ is a semi T-ABSO F. ideal, then A is a semi T-ABSO F. subm.

Proof. Let $a_s^2 x_v \subseteq A$ for F. singletons a_s of R and $x_v \subseteq X$.
Then $a_s^2 < x_v > \subseteq A$. But $< x_v > = \hat{H}X$ for some F. ideal Ĥ of R. Since X is a multiplication F. M., then $a_s^2 \hat{H} \subseteq (A:_R X)$. But $(A:_R X)$ is a semi T-ABSO F. ideal, then either $a_s \hat{H} \subseteq (A:_R X)$ or $a_s^2 \subseteq (A:_R X)$ by Proposition (2.7). Then $a_s \hat{H} X \subseteq A$ or $a_s^2 \subseteq (A:_R X)$. Thus $a_s < x_v > \subseteq A$ or $a_s^2 \subseteq (A:_R X)$. Then A is a semi T-ABSO F. subm.

Recall that "A F. M. X of an R-M $\acute{M}$ is called a cyclic F. M. if there exists $x_v \subseteq X$ such that $y_k \subseteq X$ written as $y_k = r_l x_v$ for some F. singleton r_l of R, where $k,l,v \in$ L in this case, write X=$< x_v >$ to denote the cyclic F. M. generated by x_v, [6]".

Corollary 2.10. Let A be F. subm. of cyclic F. M. X of an R- M. Ḿ. Then A is a semi T-ABSO F. subm. iff $(A:_R X)$ is a semi T-ABSO ideal.
Proof. Since every cyclic F. M. is a multiplication F. M. by[6].
By Proposition (2.8) and Proposition (2.9), then the outcome is obtained.

Recall that "If X is F. M. of an R-M.$\acute{M}$, then X is called a finitely generated F. M. if there exists $x_1, x_2, x_3, \ldots \subseteq X$ such that X=$\{a_1(x_1)_{v_1} + a_2(x_2)_{v_2} + \cdots + a_n(x_n)_{v_n}\}$, where $a_i \in R$ and $a(x)_v = (ax)_v$, $\forall v \in$L. Where $(ax)_v(y) = \begin{cases} v & if\ y = ax \\ 0 & o.w. \end{cases}$, [8]".

Recall that "If X is F. M. of an R-M. $\acute{M}$, then X is said to be a faithful F. M. if F-$annX \subseteq 0_1$ where F-$annX$=$\{x_v: r_k x_v = 0_1\ \forall\ x_v \subseteq$ X and r_k is F. singleton of R, $\forall\ v, k \in L\}$, [15]".

Corollary 2.11. Let X be a faithful finitely generated multiplication F. M. of an R- M. Ḿ and A is a proper subm. of X. Then the following expressions are equivalent:
1- A is a semi T-ABSO F. subm. of X;
2- $(A:_R X)$ is a semi T-ABSO F. ideal;
3- A=ĤX for some semi T-ABSO F. ideal Ĥ of R.
Proof. (1)⟹(2) By Proposition (2.8).
(2)⟹(3) By [6, Proposition (2.2.2)], we get the result.
(3)⟹(1) Let $r_h^2 x_v \subseteq A$ for F. singleton r_h of R and $x_v \subseteq X$, then $r_h^2 < x_v > \subseteq A$. Since X is a multiplication F. M., so that $< x_v > = KX$ for some F. ideal K of R, then $r_h^2 KX \subseteq \hat{H}X$. Since X is a faithful finitely generated multiplication F. M., hence $r_h^2 K \subseteq \hat{H}$. But Ĥ is a semi T-ABSO F. ideal, so that either $r_h K \subseteq \hat{H}$ or $r_h^2 \subseteq (\hat{H}:_R \lambda_R)$ by Proposition (2.7). Hence $r_h KX \subseteq \hat{H}X = A$ or $r_h^2 \subseteq \hat{H} = (\hat{H}X:_R X) = (A:_R X)$. Then $r_h x_v \subseteq A$ or $r_h^2 \subseteq (A:_R X)$.

Proposition 2.12. Let A be a proper F. subm. of F. M. X of an R- M. $\acute{M}$. Then the following expressions are equivalent:
1- A is a semi T-ABSO F. subm. of X;
2- $(A:_X \hat{H})$ is a semi T-ABSO F. subm. for each F. ideal Ĥ of R such that Ĥ$X \nsubseteq A$;
3- $(A:_X < a_s >)$ is a semi T-ABSO F. subm. for each F. singleton a_s of R, $a_s X \nsubseteq A$.
Proof. (1)⟹(2) Since ĤX⊈A, hence $(A:_X \hat{H}) \neq X$. Let $r_k^2 x_v \subseteq (A:_X \hat{H})$ for F. singletons r_k of R , $x_v \subseteq X$. Thus $r_k^2 \hat{H} x_v \subseteq A$. By

Proposition (2.7), either $r_k I x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$, hence $r_k x_v \subseteq (A:_X \hat{H})$ or $r_k^2 \subseteq ((A:_X \hat{H}):_R X)$.
(2)⟹(3) It is obvious.
(3)⟹(1) Since $1_v X \nsubseteq A$, hence $(A:_R < 1_v >)$ is a semi T-ABSO F. subm., then A is a semi T-ABSO F. subm. since $(A:_R < 1_v >) = A$.

Proposition 2.13. Let A be a semi T-ABSO F. subm. of F. M. X of an R- M. Ḿ. Then $(A:_R x_v)$ is a semi T-ABSO F. ideal of R, for each $x_v \subseteq X - A$.
Proof. Let $r_k^2 b_l \subseteq (A:_R x_v)$ for some F. singletons r_k, b_l of R . Hence $(r_k^2 b_l) x_v \subseteq A$**,** So that $r_k^2 (b_l x_v) \subseteq A$**.** Since A is a semi T-ABSO F. subm., then either $r_k b_l x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$, hence either $r_k b_l x_v \subseteq (A:_R x_v)$ or $r_k^2 \subseteq (A:_R X)$. Thus $(A:_R x_v)$ is a semi T-ABSO F. ideal of R.

The following proposition is a characterization of a semi T-ABSO F. subm.
Proposition 2.14. Let A be F. subm. of F. M. X of an R- M. $\acute{M}$. Then A is a semi T-ABSO F. subm. of X iff $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$ or $r_k^2 \subseteq (A:_R X)$ for each F. singletons r_k of R and $x_v \subseteq X$, ∀k,v ∈ L.
Proof. (⟹) Assume that $r_k^2 \nsubseteq (A:_R X)$. To show that $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$.
It is observe that $(A:_R r_k x_v) \subseteq (A:_R r_k^2 x_v)$. Now, let $a_s \subseteq (A:_R r_k^2 x_v)$, hence $r_k^2 a_s x_v \subseteq A$. Since A is semi T-ABSO F. subm. and $r_k^2 \nsubseteq (A:_R X)$, hence $r_k a_s x_v \subseteq A$,
so that $a_s \subseteq (A:_R r_k x_v)$. Then $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$.
(⟸) Let $r_k^2 x_v \subseteq A$, hence $(A:_R r_k^2 x_v) = \lambda_R$ where $\lambda_R(y) = \begin{cases} 1 & if\ y \in R \\ 0 & o.w. \end{cases}$
But $(A:_R r_k^2 x_v) = (A:_R r_k x_v)$ or $r_k^2 \subseteq (A:_R X)$ by hypothesis.
Thus $(A:_R r_k x_v) = \lambda_R$ and then $r_k x_v \subseteq A$. So that either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$.

Definition 2.15. Let $f: \acute{M}_1 \longrightarrow \acute{M}_2$ be a mapping and X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp., then F. kernel of a mapping f denoted by F-ker(f) is F. subm. of X_1 defined by:
F-ker(f)={x_v: $x_v \subseteq X_1$ such that $f(x_v) = 0_1$}, ∀ v∈ L.

Proposition 2.16. Let X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp. Let $f: \acute{M}_1 \longrightarrow \acute{M}_2$ be an epimorphism and A is a semi T-ABSO F. subm. of X_1 such that F- ker $f \subseteq A$. Then $f(A)$ is semi T-ABSO F. subm. of X_2.
Proof. Let $r_k^2 y_h \subseteq f(A)$ for F. singletons r_k of R and $y_h \subseteq X_2$. Since f is onto, so $y_h = f(x_v)$ for some F. singleton $x_v \subseteq X_1$, then $r_k^2 f(x_v) = f(a_s)$ for F. singleton $a_s \subseteq A$. Then $r_k^2 x_v - a_s \subseteq \mathrm{F-ker} f \subseteq A$, thus $r_k^2 x_v \subseteq A$. But A is a semi T-ABSO F. subm., hence $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X_1)$. If $r_k x_v \subseteq A$ then $r_k f(x_v) \subseteq f(a_s)$**,** hence $r_k y_h \subseteq f(A)$. If $r_k^2 \subseteq (A:_R X_1)$, then $r_k^2 X_1 \subseteq A$, hence $r_k^2 f(X_1) \subseteq f(A)$, thus $r_k^2 \subseteq (f(A):_R f(X_1))$. But $f(X_1) = X_2$ since f is onto, hence $r_k^2 \subseteq (f(A):_R X_2)$.

Remark 2.17. The condition f is an epimorphism in above proposition can't dropped, as can be proved by the following example:
Let X_1:Z→L such that $X_1(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$
Let X_2:Z→L such that $X_2(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$
It is obvious that X_1, X_2 are F. M. of Z- M. Z.
Let $f: X_1 \longrightarrow X_2$ be F. homomorphism if $f: Z \longrightarrow Z$ with $f(n) = 9n$ be homomorphism but not epimorphism, $\forall\ n \in Z$
Let A: Z→L such that $A(y) = \begin{cases} v & if\ y \in 4Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$
It is obvious that A is F. subm. of X_1.
Now, $A_v = 4Z$, $(X_1)_v = Z$ and $(X_2)_v = Z$. $A_v = 4Z$ is a semi T-ABSO subm., but $f(4Z) = 36Z$ is not semi T-ABSO since 2^2.9∈36Z ,but 2^2∉36Z and 2. 9∉36Z. So that A isa semi T-ABSO F. subm., but $f(A)$ is not semi T-ABSO F. subm.

Proposition 2.18. Let X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp. Let $f: \acute{M}_1 \longrightarrow \acute{M}_2$ be an epimorphism, B is a semi T-ABSO F. subm. of X_2 . Then $f^{-1}(B)$ is a semi T-ABSO F. subm. of X_1.
Proof. Let $r_k^2 x_v \subseteq f^{-1}(B)$ for F. singletons r_k of R and $x_v \subseteq X_1$, hence $f(r_k^2 x_v) \subseteq B$ so $r_k^2 f(x_v) \subseteq B$. Since B is semi T-ABSO F. subm., then either $r_k f(x_v) \subseteq B$ or $r_k^2 \subseteq (B:_R X_2)$, so that $r_k x_v \subseteq f^{-1}(B)$ or $r_k^2 \subseteq (B:_R X_2)$.
If $r_k^2 \subseteq (B:_R X_2)$, then $r_k^2 X_2 \subseteq B$, hence $r_k^2 f(X_1) \subseteq B$. So that $r_k^2 X_1 \subseteq f^{-1}(B)$. Then $r_k^2 \subseteq (f^{-1}(B):_R X_1)$. So that either $r_k x_v \subseteq f^{-1}(B)$ or $r_k^2 \subseteq (f^{-1}(B):_R X_1)$.

Recall that "A F. ideal K of a ring R is called a principle F. ideal if there exists $x_v \subseteq K$ such that $K = \langle x_v \rangle$. For each $a_s \subseteq K$, there exists F. singleton b_l of R such that $a_s = b_l x_v$ where v,s,l ∈ L, that is $K = \langle x_v \rangle = \{a_s \subseteq K: a_s = b_l x_v$ for some F. singleton b_l of R}, [10]".

Proposition 2.19. Let R be a principle F. ideal ring (P. F.I.R) and X be F. M. of an R- M. Ḿ. Let A be a proper F. subm. of X and Ĥ be F. ideal of R . Then A is a semi T-ABSO F. subm. of X iff $\hat{H}^2 x_v \subseteq A$ implies$\hat{H} x_v \subseteq A$ or $\hat{H}^2 \subseteq (A:_R X)$ for any F. ideal Ĥ of R and F. singleton $x_v \subseteq X$.
Proof. (⇒) Suppose that Ĥ be F. ideal of R and F. singleton $x_v \subseteq X$. Since R is P. F.I.R , hence $\hat{H} =< r_k >$ for some F. singleton r_k of R. If $\hat{H}^2 x_v \subseteq A$ then $< r_k >^2 x_v \subseteq A$, thus $r_k^2 x_v \subseteq A$, then either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X)$. Hence $\hat{H} x_v \subseteq A$ or $\hat{H}^2 \subseteq (A:_R X)$
(⇐) It is obvious.

Recall that "Let A and B be two F. subms of F. M. X. If X=A+B and A∩B=0_1, then X is called F. internal direct sum of A and B and denoted by A⊕B. Define by:
(A⊕B)(a,b)=min{A(a), B(b) for all (a,b) ∈ $M_1 \oplus M_2$}
Moreover, A and B are called direct summand of X, [6]".

Proposition 2.20. Let $X = X_1 \oplus X_2$ be F. M. of an R- M. $\acute{M} = \acute{M}_1 \oplus \acute{M}_2$ where X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp.. Let A, B be proper F. subms of X_1, X_2 resp., then
1- A is semi T-ABSO F. subm. in X_1 iff $A \oplus X_2$ is semi T-ABSO F. subm. in $X_1 \oplus X_2 = X$.
2- B is semi T-ABSO F. subm. in X_2 iff $X_1 \oplus B$ is semi T-ABSO F. subm. in $X_1 \oplus X_2 = X$.
Proof. (1) (⇒) Let $r_k^2 (x_v, y_h) \subseteq A \oplus X_2$ for F. singletons r_k of R and $(x_v, y_h) \subseteq X$**.** Hence $r_k^2 x_v \subseteq A$ and $r_k^2 y_h \subseteq X_2$. Since A is semi T-ABSO F. subm. in X_1, then either $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A:_R X_1)$. So that $r_k (x_v, y_h) \subseteq A \oplus X_2$ or $r_k^2 \subseteq (A \oplus X_2:_R X_1 \oplus X_2)$. Then $A \oplus X_2$ is semi T-ABSO F. subm. in $X_1 \oplus X_2 = X$.
(⇐) Let $r_k^2 x_v \subseteq A$ for F. singletonsr_k of R and $x_v \subseteq X_1$, hence for any F. singleton $y_h \subseteq X_2$, $r_k (x_v, y_h) \subseteq A \oplus X_2$. Since $A \oplus X_2$ is a

semi T-ABSO F. subm.. in X, then either $r_k(x_v, y_h) \subseteq A \oplus X_2$ or $r_k^2 \subseteq (A \oplus X_2 :_R X_1 \oplus X_2) = (A :_R X_1)$. So that $r_k x_v \subseteq A$ or $r_k^2 \subseteq (A :_R X_1)$. Then A isa semi T-ABSO F. subm. in X_1.
(2) The proof by the same method in (1).

Proposition 2.21. Let X_1, X_2 be F. M. of an R M. $\acute{M}_1, \acute{M}_2$ resp. and $X = X_1 \oplus X_2$ be F. M. of an R- M. $\acute{M} = \acute{M}_1 \oplus \acute{M}_2$ such that $F - annX_1 \oplus F - annX_2 = \lambda_R$ where $\lambda_R(y) = 1$, ∀y∈ R. Let A be a semi T-ABSO F. subm. of X, then either
1- $A = A_1 \oplus X_2$ and A_1 is a semi T-ABSO F. subm. in X_1 or
2- $A = X_1 \oplus A_2$ and A_2 is a semi T-ABSO F. subm. in X_2 or
3- $A = A_1 \oplus A_2$ and A_1 is a semi T-ABSO F. subm. in X_1 and A_2 is a semi T-ABSO F. subm. in X_2.
Proof. Since $f - annX_1 \oplus f - annX_2 = \lambda_R$ where $\lambda_R(y) = 1$, ∀y∈ R , then by [5], $A = A_1 \oplus A_2$ for some F. subm. A_1 of X_1 and A_2 of X_2 . Then we have:
(1) $A_1 < X_1$ and $A_2 = X_2$.
(2) $A_1 = X_1$ and $A_2 < X_2$.
(3) $A_1 < X_1$ and $A_2 < X_2$.
Case(1) and case(2), we get $A = A_1 \oplus X_2$ or $A = X_1 \oplus A_2$. Then A_1 is semi T-ABSO F. subm. in X_1 or A_2 is semi T-ABSO F. subm. in X_2 by Proposition (2.20).
Case(3): Suppose that $r_k^2 x_v \subseteq A$ for F. singletons r_k of R and $x_v \subseteq X_1$. Hence $r_k^2(x_v, 0_1) \subseteq A_1 \oplus A_2 = A$. But A be a semi T-ABSO F. subm. of X, then either $r_k(x_v, 0_1) \subseteq A$ or $r_k^2 \subseteq (A :_R X) \subseteq (A_1 :_R X_1)$ implies that $r_k x_v \subseteq A_1$ or $r_k^2 \subseteq (A_1 :_R X_1)$. Then A_1 is a semi T-ABSO F. subm. in X_1.
By the same method we get A_2 is a semi T-ABSO F. subm.. in X_2

"**Definition 2.22.** A F. M. X of an R- M. M is called a duo F. M. if for each F. subm. A of X, A is fully invariat, [5]"**.**
Note:If $X = X_1 \oplus X_2$ be F. M. of an R- M. $\acute{M} = \acute{M}_1 \oplus \acute{M}_2$ is a duo F. M. or a distributive F. M. see[9], we can have the same inference of Proposition (2.21).

Proposition 2.23. Let X_1, X_2 be F. M. of an R- M. $\acute{M}_1, \acute{M}_2$ resp. and A_1, A_2 are semi T-ABSO F. subms of X_1, X_2 resp. such that $(A_1 :_R X_1) = (A_2 :_R X_2)$. Then $A = A_1 \oplus A_2$ is a semi T-ABSO F. subm. of $X = X_1 \oplus X_2$.
Proof. Let $r_k^2(x_v, y_h) \subseteq A_1 \oplus A_2$, so that $r_k^2 x_v \subseteq A_1$ and $r_k^2 y_h \subseteq A_2$. Since A_1, A_2 are semi T-ABSO F. subms, then $r_k x_v \subseteq A_1$ or $r_k^2 \subseteq (A_1 :_R X_1)$ and $r_k y_h \subseteq A_2$ or $r_k^2 \subseteq (A_2 :_R X_2) = (A_1 :_R X_1)$, hence $r_k x_v \subseteq A_1$ and $r_k y_h \subseteq A_2$ or $r_k^2 \subseteq (A_1 :_R X_1)$.
Then $r_k(x_v, y_h) \subseteq A_1 \oplus A_2$ or $r_k^2 \subseteq (A :_R X)$. Thus A is a semi T-ABSO F. subm. of X.

3. Semi T-ABSO F. M.

In this section we present the concept of semi T-ABSO F. M. Some of properties and relationships with other classes of F. M. are illustrated.
First, we give the following definition.
Definition 3.1. A F. M. X of an R- M. $\acute{M}$ is called T-ABSO F. M. if the zero F. subm. (0_1) is T-ABSO F.; that is if for each F. singleton a_s, b_l of R and $x_v \subseteq X$, ∀s,l,v ∈ L, such that $a_s b_l x_v = 0_1$ implies $a_s x_v = 0_1$ or $b_l x_v = 0_1$ or $a_s b_l \subseteq F - annX$.

Now, we present the concept of a semi T-ABSO F. M. as follows:

Definition 3.2. Let X be F. M. of an R- M. $\acute{M}$, X is called a semi T-ABSO F. M. if 0_1 is a semi T-ABSO F. subm. of X.

The proposition specificities a semi T-ABSO F. M. in terms of its level M. is given:
Proposition 3.3. Let X be F. M. of an R- M. $\acute{M}$. Then X is a semi T-ABSO F. M. iff the level X_v is a semi T-ABSO M., ∀ v∈ L.
Proof. (⟹) Let $a^2 x = 0$ for each a∈ R , $x \in X_v$, ∀ v∈ L, then $(a^2 x)_v \subseteq 0_v \subseteq 0_1$, hence $a_s^2 x_k \subseteq 0_1$ where v=min{s, k} and $(a^2)_s = a_s^2$. But 0_1 is a semi T-ABSO F. subm. by Definition (3.2), then either $a_s x_k \subseteq 0_1$ or $a_s^2 \subseteq (0_1 :_R X) = F - annX$, hence $(ax)_v \subseteq 0_1$ or $(a^2)_v \subseteq F - annX$, implies $ax = 0$ or $a^2 \in annX_v$. Then (0) is a semi T-ABSO subm. of X_v.
(⟸) Let $a_s^2 x_k \subseteq 0_1$ for F. singleton a_s of R and $x_v \subseteq X$, then $(a^2 x)_v \subseteq 0_1$ where v=min{s, k}, hence $0_1(a^2x) \geq v$. If $a^2x \neq 0$, then $0_1(a^2x)=0 \geq v$ which is a contradiction. so that $a^2x=0$. But (0) is a semi T-ABSO subm. of X_v, then either $ax = 0$ or $a^2 \in annX_v$, hence $(ax)_v \subseteq 0_1$ or $(a^2)_v \subseteq F - annX$, so that $a_s x_k \subseteq 0_1$ or $a_s^2 \subseteq F - annX$. Thus 0_1 is semi T-ABSO F. subm. of X.

Remarks and Examples 3.4.
(1) Every semiprime F. M. is a semi T-ABSO F. M., but the converse incorrect, for example:
Let X:Z_{49}→L such that $X(y) = \begin{cases} 1 & if\ y \in Z_{49} \\ 0 & o.w. \end{cases}$
It is obvious that X be F. M. of Z- M. Z_{49}.
$X_v = Z_{49}$ as Z- M. is a semi T-ABSO M. since $7^2.\bar{1} = 0$ implies $7^2 \in (0 :_Z Z_{49}) = 49Z$, but X_v is not semiprime M. since 7. $\bar{1}$≠0 . So that X is a semi T-ABSO F. M., but it is not semiprime F. M. by [12].
(2) Every T-ABSO F. M. is a semi T-ABSO F. M.
(3) Every quasi-prime F. M. is a semi T-ABSO F. M. But the converse incorrect see the example in part(1) where $X_v = Z_{49}$ as Z- M. is semi T-ABSO M., but X_v is not quasi-prime M. since 7.7. $\bar{1}$=0 and 7. $\bar{1}$≠0, So that X is semi T-ABSO F. M., but it is not quasi-prime F. M. by [6].
(4) Every F. subm. of a semi T-ABSO F. M. is a semi T-ABSO F. M.

Proposition 3.5. Let X be F. M. of an R- M. $\acute{M}$. If X is a semi T-ABSO F. M., then $F - ann_R X$ is semi T-ABSO F. ideal.
Proof. Since X is semi T-ABSO F. M., then 0_1 is semi T-ABSO F. subm. By Proposition (2.8) when A=0_1**,** we have $(0_1 :_R X) = F - ann_R X$ is a semi T-ABSO F. ideal.

Proposition 3.6. Let X be a multiplication F. M.of an R- M. $\acute{M}$. Then X is a semi T-ABSO F. M. iff $F - ann_R X$ is a semi T-ABSO F. ideal.
Proof. (⟹) By Proposition (3.5)**,** we get the outcome.
(⟸)By Proposition (2.9)**,** we get the outcome.

Corollary 3.7. Let X be a faithful multiplication F. M. of an R- M. $\acute{M}$. Then the following expressitions are equivalent:
1- X is a semi T-ABSO F. M.;
2- R is a semi T-ABSO F. ring.
Proof. (1) Since X is a semi T-ABSO F. M., so that $F - ann_R X$ is semi T-ABSO F. ideal by Proposition (3.6). But $F - ann_R X = 0_1$, hence 0_1 is semi T-ABSO F. ideal.
Then R is semi T-ABSO F. ring.

(2) Since R is a semi T-ABSO F. ring, so that 0_1 is semi T-ABSO F. ideal, but $0_1 = F - ann_R X$ since X is a faithful. Then X is semi T-ABSO F. M. by Proposition (3.6).

Proposition 3.8. Let X be F. M. of an R- M. Ḿ such that $F - ann_R X$ is a semiprime F. ideal of R . Then **X** is semi T-ABSO F. M. iff X is semiprime F. M.

Proof. (⟹) Let $r_k^2 x_v \subseteq 0_1$ for F. singletons r_k of R and $x_v \subseteq X$. Since X is semi T-ABSO F. M., then $r_k x_v \subseteq 0_1$ or $r_k^2 \subseteq (0_1 :_R X) = F - ann_R X$. Hence $r_k x_v \subseteq 0_1$ or $r_k \subseteq F - ann_R X$ since $F - ann_R X$ is semiprime F. ideal of R. Thus $r_k x_v \subseteq 0_1, \forall x_v \subseteq X$.Then 0_1 is semiprime F. subm.. So that X is semiprime F. M. by [11].

(⟸) It is obvious.

Proposition 3.9. Let X be F. M. of an R- M. $\acute{M}$. If X is a semi T-ABSO F. M., then $F - ann_R A$ is semi T-ABSO F. ideal for each non-constant F. subm. A of X.

Proof. Let A be a non-empty F. subm. of X and $F - ann_R A \neq \lambda_R$ because if $F - ann_R A = \lambda_R$, then $A = 0_1$ which is a contradiction. Now, suppose that $r_k^2 a_s \subseteq F - ann_R A$ for F. singletons r_k, a_s of R. Hence $r_k^2 a_s A \subseteq 0_1$. Since **X** is semi T-ABSO F. M., then either $r_k a_s A \subseteq 0_1$ or $r_k^2 \subseteq (0_1 :_R X)$ by Proposition (2.7). Hence either $r_k a_s \subseteq F - ann_R A$ or $r_k^2 \subseteq F - ann_R A$ since $F - ann_R X \subseteq F - ann_R A$ by [6]. Thus $F - ann_R A$ is semi T-ABSO F. ideal.

Recall that "A ring R is said to be an integral domain if R has no zero-divisor F. singleton (i.e. if a_v is F. singleton of R $\exists b_l$ is F. singleton of R such that $a_v b_l = 0_1$, $\forall v$, $l \in$L, implies $a_v = 0_1$ or $b_l = 0_1$), [16]".

Recall that " A F. subm A of F. M. X is called a divisible F. if for each F. singleton $x_v \subseteq A$ there exists F. singleton $y_h \subseteq A$ and for each $r \in$ R, $r \neq 0$, $x_v = r y_h$ where $(ry)_h = r y_h$, X is called a divisible F. M. if X is F. divisible subm. of itself, [14]".

Proposition 3.10. Let R is an integral domain and X is a non-empty divisible F. M. of an R- M. Ḿ. Then X is semi T-ABSO F. M. iff X is quasi-prime F. M

Proof. (⟹) Let $r_k a_s x_v \subseteq 0_1$ for F. singletons r_k, a_s of R and $x_v \subseteq X$.

If $r_k a_s \subseteq 0_1$, then $r_k \subseteq 0_1$ or $a_s \subseteq 0_1$, so that $r_k x_v \subseteq 0_1$ or $a_s x_v \subseteq 0_1$.

If $r_k a_s \nsubseteq 0_1$, then $r_k \nsubseteq 0_1$ or $a_s \nsubseteq 0_1$ since R is an integral domain.

If $r_k x_v \subseteq 0_1$, then the proof is complete.

If $r_k x_v \nsubseteq 0_1$, $r_k \nsubseteq 0_1$ and X is a divisible F. M., hence $r_k X = X$, then $x_v = r_k y_h$ for F. singleton $y_h \subseteq X$, thus $r_k a_s x_v = r_k a_s r_k y_h = r_k^2 a_s y_h \subseteq 0_1$. But 0_1 is semi T-ABSO F. subm., then either $r_k a_s y_h \subseteq 0_1$ or $r_k^2 \subseteq F - ann_R X$. If $r_k^2 \subseteq F - ann_R X$ then $r_k^2 X \subseteq 0_1$, but $r_k \nsubseteq 0_1$hence$r_k^2 \nsubseteq 0_1$. Then $r_k^2 X = X \subseteq 0_1$ this is a contradiction. Thus $r_k^2 \nsubseteq F - ann_R X$, then $r_k a_s y_h \subseteq 0_1$ so that $a_s x_v \subseteq 0_1$. Thus 0_1 is quasi-prime F. subm.

(⟸) It is obvious.

Corollary 3.11. Let R be an integral domain and X is a non-empty divisible F. M. of an R- M. Ḿ. Then the following expressions are equivalent:

(1) X is a semi T-ABSO F. M.

(2) X is a quasi-prime F. M.

(3) X is a prime F. M.

Proof. (1)⟺(2) It follows by Proposition(3.10).

(2)⟺(3) It follows by [6]**.**

(3)⟺(1) It follows by [11 , 6]and Proposition(3.10).

Proposition 3.12. A F. M. X of an R- M. Ḿ is a semi T-ABSO F. M. iff either $F - ann\, r_k x_v = F - ann\, r_k^2 x_v$ for any F. singletons r_k of R and $x_v \subseteq X$ such that $r_k x_v \nsubseteq 0_1$ or $r_k^2 X \subseteq 0_1$.

Proof. (⟹) Let $a_s \subseteq F - ann\, r_k^2 x_v$, $r_k^2 x_v \nsubseteq 0_1$.
Then $r_k^2 a_s x_v \subseteq 0_1$. But X is a semi T-ABSO F. M. and $r_k^2 \nsubseteq F - ann\, X$, hence $r_k a_s x_v \subseteq 0_1$, so that $a_s \subseteq F - ann\, r_k x_v$. Then $F - ann\, r_k x_v = F - ann\, r_k^2 x_v$.

(⟸) It is obvious.

Proposition 3.13. Let $X = X_1 \oplus X_2$ be F. M. of an R- M. $Ḿ = Ḿ_1 \oplus Ḿ_2$. If X is semi T-ABSO F. M., then X_1 and X_2 are semi T-ABSO F. M.

Proof. By Remarks and Examples(3.4) part(4) the outcome hold.

Remark 3.14. The converse of Proposition(3.13) is not true always, for example:

Let X:$Z_2 \oplus Z_{49}$→L such that X(x,y)= $\begin{cases} 1 & if\ (x,y) \in Z_2 \oplus Z_{49} \\ 0 & o.w. \end{cases}$

It is obvious that X be F. M. of Z- M. $Z_2 \oplus Z_{49}$.

And $X_1 : Z_2 \longrightarrow$ L such that $X_1(x) = \begin{cases} 1 & if\ x \in Z_2 \\ 0 & o.w. \end{cases}$

It is obvious that X_1 be F. M. of Z- M. Z_2.

$X_2 : Z_{49} \longrightarrow$ L such that $X_2(y) = \begin{cases} 1 & if\ y \in Z_{49} \\ 0 & o.w. \end{cases}$

It is obvious that X_2 be F. M. of Z- M. Z_{49}.

Now, $X_v = Z_2 \oplus Z_{49}$ as Z- M. where $(X_1)_v = Z_2$ and $(X_2)_v = Z_{49}$ are semi T-ABSO M., but $X_v = Z_2 \oplus Z_{49}$ is not semi T-ABSO M. since $7^2(\bar{0},\bar{1}) = (\bar{0},\bar{0})$, but $7(\bar{0},\bar{1}) = (\bar{0},\bar{7}) \neq (\bar{0},\bar{0})$ and $7^2 \notin annX_v = annZ_2 \cap annZ_{49} = 2Z \cap 49Z = 98Z$. So that X_1 and X_2 are semi T-ABSO F. M. but X is not semi T-ABSO F. M.

Theorem 3.15. Let $X = X_1 \oplus X_2$ be F. M. of an R- M. $Ḿ = Ḿ_1 \oplus Ḿ_2$ where X_1 and X_2 be prime F. M. Then $X = X_1 \oplus X_2$ is semi T-ABSO F. M.

Proof. Let $r_k^2(x_v, y_h) \subseteq (0_1, 0_1)$ for F. singletons r_k of R and $(x_v, y_h) \subseteq X$. Hence $r_k^2 x_v \subseteq 0_1$ and $r_k^2 y_h \subseteq 0_1$, then $r_k(r_k x_v) \subseteq 0_1$ and $r_k(r_k y_h) \subseteq 0_1$. Since X_1 and X_2 be a prime F. M., then either($r_k x_v \subseteq 0_1$ or $r_k \subseteq F - annX_1$) and ($r_k y_h \subseteq 0_1$ or $r_k \subseteq F - annX_2$)

Then there exist four case:

1) If $r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$, then $r_k(x_v, y_n) \subseteq 0_1$.

2) If $r_k \subseteq F - ann\, X_1$ and $r_k \subseteq F - ann\, X_2$, then $r_k \subseteq F - ann\, X_1 \cap F - annX_2 = F - ann\, X$, but $r_k \subseteq F - ann\, X$ implies $r_k^2 \subseteq F - annX$.

3) If $r_k x_v \subseteq 0_1$ and $r_k \subseteq F - ann\, X_2$, then $r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$, hence $r_k\, (x_v, y_h) \subseteq 0_1$.

4) If $r_k \subseteq F - ann\, X_1$ and $r_k y_h \subseteq 0_1$, then $r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$, hence $r_k(x_v, y_h) \subseteq 0_1$. Then X is a semi T-ABSO F. M.

Remarks 3.16.

(1) By an application of Theorem(3.15), each of the following F. M. is a semi T-ABSO F. M. of an R- M. Z.

$X:Z_p\oplus Z_q\longrightarrow L$, $X:Z_p\oplus Z_p\longrightarrow L$, $X:Z_p\oplus Z\longrightarrow L$, $X:Q\oplus Z\longrightarrow L$, $X:Z\oplus Z\longrightarrow L$ and $X:Q\oplus Q\longrightarrow L$ where p, q are two prime numbers.
(2) The condition X_1 and X_2 be prime F. M. can't deleted from Theorem (3.15), see Remarks (3.14) where $X_v = Z_2\oplus Z_{49}$ as Z- M. , $(X_1)_v = Z_2$ as Z- M. is a prime M. and $(X_2)_v = Z_{49}$ as Z- M. is not prime M. also $X_v = Z_2\oplus Z_{49}$ is not semi T-ABSO M., then X_1 is prime F. M. , X_2 is not prime F. M. and X is not semi T-ABSO F. M.

Proposition 3.17. Let $X = X_1\oplus X_2$ be F. M. of an R- M. $\acute{M} = \acute{M}_1\oplus\acute{M}_2$ such that $F - annX_1 = F - annX_2$. Then X is semi T-ABSO F. M. iff X_1 and X_2 are semi T-ABSO F. M.
Proof. ($\Longleftarrow$) Let $r_k^2(x_v, y_h) \subseteq (0_1, 0_1)$ for F. singletons r_k of R and $(x_v, y_h) \subseteq X$.
Hence $r_k^2 x_v \subseteq 0_1$ and $r_k^2 y_h \subseteq 0_1$. Since X_1 and X_2 be a semi T-ABSO F. M., then either($r_k x_v \subseteq 0_1$ or $r_k^2 \subseteq F - annX_1$) and ($r_k y_h \subseteq 0_1$ or $r_k^2 \subseteq F - annX_2 = F - annX_1$). Thus ($r_k x_v \subseteq 0_1$ and $r_k y_h \subseteq 0_1$) or $r_k^2 \subseteq F - annX_1$. Then $r_k(x_v, y_h) \subseteq (0_1, 0_1)$ or $r_k^2 \subseteq F - annX_1 = F - annX_1 \cap F - annX_2 = F - annX$.
So that X is semi T-ABSO F. M.
($\Longleftarrow$) It is obvious.

Remarks 3.18. The condition $F - annX_1 = F - annX_2$ is obligate for Proposition (3.17), so we can't dropped it, we see the following example:
Let $X:Z_9\oplus Q \longrightarrow L$ such that $X(x,y)=\begin{cases}1 & if\ (x,y) \in Z_9\oplus Q\\ 0 & o.w.\end{cases}$
It is obvious that X be F. M. of Z- M. $Z_9\oplus Q$.
And $X_1: Z_9 \longrightarrow L$ such that $X_1(x) = \begin{cases}1 & if\ \ x \in Z_9\\ 0 & o.w.\end{cases}$
It is clear that X_1 is F. M. of Z- M. Z_9.
$X_2: Q \longrightarrow L$ such that $X_2(y) = \begin{cases}1 & if\ \ y \in Q\\ 0 & o.w.\end{cases}$
It is obvious that X_2 be F. M. of Q as Z- M.
Now, $X_v = Z_9\oplus Q$ as Z- M. and $(X_1)_v = Z_9$ as Z- M. ,
$(X_2)_v = Q$ as Z- M., where $X_v = Z_9\oplus Q$ is not semi T-ABSO M. since $3^2(\bar{1},\bar{0}) = (\bar{0},\bar{0})$, but $3(\bar{1},\bar{0}) \neq (\bar{0},\bar{0})$ and $3^2 \notin annX_v = ann_Z Z_9 \cap ann_Z Q = 0$, but each of $(X_1)_v = Z_9$ as Z- M. , $(X_2)_v = Q$ as Z- M. is a semi T-ABSO M. and $ann_Z Z_9 = 9Z \neq ann_Z Q = 0$. So that X is not semi T-ABSO F. M., but X_1 and X_2 are semi T-ABSO F. M. and $F - annX_1 \neq F - annX_2$.
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Proposition 3.19. The following expressions are equivalent for F. M. X of an R- M. $\acute{M}$
(1) X is a semi T-ABSO F. M.
(2) $F - ann_X\hat{H}$ is a semi T-ABSO F. subm. for each F. ideal $\hat{H}$ of R with $\hat{H} \nsubseteq F - annX$.
(3) $F - ann_X < a_s >$ is a sem T-ABSO F. subm. for each F. singleton a_s of R with $a_s \nsubseteq F - annX$, ∀s∈ L.
Proof. It follows by Proposition (2.12) with $A=0_1$.

Now, we give the concept of a comultiplication F. M. as follows:
Definition 3.20. A F. M. X of an R-M. $\acute{M}$ is called a comultiplication F. M. if $A = F - ann_X F - ann_R\, A$ for each F. subm. A of X.
Proposition 3.21. If X is a semi T-ABSO comultiplication F. M. of an R- M. $\acute{M}$. Then every proper F. subm. of X is a semi T-ABSO F. subm.

Proof. Let A be a proper F. subm. of X, hence $A = F - ann_X F - ann_R A$. Put $F - ann_R A = \hat{H}$, so that $A = F - ann_X\hat{H}$. But $\hat{H} \nsubseteq F - ann_R X$ since if $\hat{H} \subseteq F - ann_R X$ hence $F - ann_R X = F - ann_R A$ and then $A = X$ which is a contradication.
Then by Proposition (3.18), $A = F - ann_X\hat{H}$ is semi T-ABSO F. subm. Hence every proper F. subm. A of X is semi T-ABSO F

References

[1] Abdulrahman A.H.,*2-Absorbing Submodules(Modules) and Some of Their Generalizations*. M.Sc. Thesis, University of Baghdad, 2015.
[2] Darani, A.Y. and Soheilnia, F., 2-Absorbing and Weakly 2-Absorbing Submodules,*Thai Journal of Mathematics*, Vol.9, No.3, 2011, pp.577-584.
[3] Deniz S., Gürsel Y., Serkan O., Bayram A. E. and Bijan D. **,** On 2-Absorbing Primary Fuzzy Ideals of Commutative Rings, *Hindawi, Mathematical Problems in Engineering*, Vol.2017, 2017, pp. 1-7.
[4] Hadi, I. M.A., Semiprime Fuzzy Submodules of Fuzzy Modules, *Ibn-Al-Haitham J. for Pure and Appl. Sci.*,Vol.17, No.3, 2004, pp.112-123.
[5] Hadi, G. Rashed, *Fully Cancellation Fuzzy Modules and Some Generalizations*. M.Sc. Thesis, University of Baghdad, 2017.
[6] Hatam,Y. K., *Fuzzy Quasi-Prime Modules and Fuzzy Quasi-Prime Submodules*, M.Sc. Thesis, University of Baghdad, 2001.
[7] Hatam Y. Khalaf and Wafaa H. Hannon, Small and Classical T-ABSO Fuzzy Submodules, *Global Journal of Pure and Applied Mathematics*, Vol.14 No.3, 2018, pp. 443–457.
[8] Inaam, M.A. Hadi and Maysoun, A. Hamil.,Cancellation and Weakly Cancellation Fuzzy Modules" *Journal of Basrah Reserches ((Sciences))* , Vol.37, No. 4, 2011.
[9] Inaam, M. A. Hadi and Shrooq. B.Semeein, Fuzzy Distributive Modules, *Ibn AL- Haitham J. for Pure & Appl. Sci.*, Vol.24, No.1, 2011.
[10] Martinez,L., Fuzzy Modules Over Fuzzy Rings in Connection With Fuzzy Ideals of Rings, *J. Fuzzy Math.*, Vol.4, 1996, pp.843-857.
[11] Maysoun A. H., Semiprime Fuzzy Modules, *Ibn Al-Haitham Journal for Pure and Applied Science*, Vol.25, No.1, 2012.
[12] Mohammed M. R. AL-Shamiri, On Fuzzy Semiprime Submodules, *International Journal of Innovation and Applied Studies*, Vol.13, No.4, 2015, pp.929-934.
[13] Mukhrjee, T. K., Prime Fuzzy Ideals in Rings, *Fuzzy Sets and Systems*, Vol.32, 1989, pp.337-341.
[14] Qaid, A. A., *Some Results on Fuzzy Modules*. M.Sc., Thesis, University of Baghdad, 1999.
[15]Rabi H.J., *Prime Fuzzy Submodules and Prime Fuzzy Modules*. M.Sc. Thesis, University of Baghdad, 2001.
[16] Wafaa R. H., (1999), "Some Results of Fuzzy Rings", M.Sc. Thesis University of Baghdad.
[17] Zahedi, M. M, On L-Fuzzy Residual Quotiet Modules and P-Primary Submodules, *Fuzzy Sets and Systems*, Vol.51, 1992, pp. 333-344.