On μ*-extending modules

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Abstract: Let R be an associative ring with identity and let M be a left R- module. As a generalization of essential submodules Zhou defined an Ғ- essential submodules provided it has a nonzero intersection with any nonzero submodule in Ғ where Ғ is a collection of R- modules such that if $M \in$ Ғ , then $M' \in$ Ғ for any module M' isomorphic to M. In this article we study μ*- essential submodules as a dual of μ-small submodules provided it has a nonzero intersection with any nonzero singular submodule of M. Also we define and investigate μ*-extending modules with some examples and basic properties.

Keywords. μ*-essential, μ*-closed submodules, μ*-extending modules.

1. Introduction

Let R be an associative ring with unity and let M be unitary left R- module. A submodule A of M is said to be essential in M, (denoted by $A \leq_e M$), if for any submodule B of M , $A \cap B = 0$ implies $B = 0$ [1], and a submodule A of M is said to be closed in M if A has no proper essential extension in M; that is if $A \leq_e B \leq M$, then $A = B$ [1]. An R- module M is called extending (or CS- module) , if every submodule of M is essential in a direct summand of M. It is well known that an R- module M is extending if and only if every closed submodule of M is a direct summand [2]. A submodule A of M is called μ- small submodule of M (denoted by $A \ll_\mu M$) if whenever $M = A + X$, $\frac{M}{X}$ is cosingular , then $M = X$, see [3]

Following [4], Zhou defined an Ғ- essential submodules provided it has a nonzero intersection with any nonzero submodule in Ғ where Ғ is a collection of R- modules such that if $M \in$ Ғ , then $M' \in$ Ғ for any module M' isomorphic to M. In this paper we introduce μ*- essential submodules as a dual of μ-small submodules provided it has a nonzero intersection with any nonzero singular submodule of M.

An R- module M is called μ*- extending module if every submodule of M is μ*- essential in a direct summand.

In section two , we define and study μ*-essential submodules , μ*- closed submodules and μ*- uniform modules.

In section three , we introduce μ*- extending modules with some examples and basic properties , we give sufficient conditions for a submodules of μ*- extending modules to be μ*- extending module.

In section four , we give various characterizations of μ*- extending modules and study the direct sum of μ*- extending modules.

2. μ*-essential and μ*- closed submodules.

In this section, we introduce μ*- essential submodules and μ*- uniform modules as a generalization of essential submodules and uniform modules respectively which are duals of μ- small submodules and μ- hollow modules. Also , we define a μ*- closed submodules which is stronger than closed submodules. We study the basic properties of them that are relevant to our work.

Definition (2.1): Let A be a submodule of an R- module M, M is said to be **μ*-essential extension** to A or A is a **μ*-essential in M** if for any nonzero singular submodule B of M , we have $A \cap B \neq 0$. It will be denoted by $A \leq_{\mu^* e} M$.

Remarks and Examples (2.2).

(1) It is clear that μ*- essential submodules are generalizations of essential submodules. There is a μ*-essential submodule of an R- module M which is not essential in M. For example: Consider Z_6 as Z_6- module . Since Z_6 is nonsingular Z_6- module , then $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are μ*- essential in Z_6 which are not essential in Z_6.

(2) Every nonzero submodule of Q as Z- module is μ*-essential in Q.

(3) Every nonzero cyclic submodule of Z as Z- module is μ*- essential in Z.

(4) Consider Z_6 as Z- module , $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are not μ*- essential in Z_6.

In the following propositions we consider conditions under which μ*-essential submodules versus essential submodules.

Proposition(2.3): Let M be a singular R- module and let A be a submodule of M , then $A\leq_{\mu^*e} M$ if and only if $A\leq_e M$.

Proof: It is clear.

Let R be a commutative integral domain and M be an R-module. Recall that $T(M) = \{m\in M: rm = 0$, for some nonzero $r\in R\}$ is called the torsion submodule of M. If $T(M) = M$ (if $T(M) = 0$), then M is called **torsion (torsion free) module**, see [5].

Proposition (2.4): Let M be a torsion module over a commutative integral domain R and A be a submodule of M. Then $A\leq_{\mu^*e} M$ if and only if $A\leq_e M$.

Proof: It is clear by [5, P. 31] and Prop. (2.3).

Let M be an R-module . Recall that M is called a **prime** R- module if $ann(x) = ann(y)$, for every nonzero elements x and y in M , see [6].

Proposition (2.5): Let M be a prime R- module with $Z(M)\neq 0$ and A be a submodule of M. Then $A\leq_{\mu^*e} M$ if and only if $A\leq_e M$.

Proof: Assume that $A\leq_{\mu^*e} M$. To show that M is singular . Let $0\neq x\in Z(M)$, then $ann(x)\leq_e R$ and let $0\neq y\in M$. Since M is prime module , then $ann(x) = ann(y)$ and hence $y\in Z(M)$. Thus $Z(M) = M$ and hence $A\leq_e M$, by Prop. (2.3). The proof of the converse is clear. □

Next, we give characterizations of μ*- essential submodules.

Proposition (2.6): Let M be an R- module and let A be a submodule of M , then $A\leq_{\mu^*e} M$ if and only if for any nonzero cyclic singular submodule K of M , $A\cap K\neq 0$.

Proof: Let K be a nonzero cyclic singular submodule of M and let $0\neq x\in K$. By our assumption $0\neq \langle x\rangle\cap A\leq A\cap K$. Hence $A\cap K\neq 0$. The proof of the converse is clear. □

Proposition (2.7): Let M be an R- module and let A be a submodule of M , then $A\leq_{\mu^*e} M$ if and only if for any nonzero element x in M with Rx singular has a nonzero multiple in A.

Proof: Let $0\neq x\in M$ with Rx singular submodule of M. By Prop. (2.6) $Rx\cap A\neq 0$. Hence there is $r\in R$ such that $0\neq rx\in A$. The proof of the converse is clear. □

Proposition (2.8): Let M be any R- module. Then the following are hold.

(1) Let submodules $A\leq B\leq M$. Then $A\leq_{\mu^*e} M$ if and only if $A\leq_{\mu^*e} B$ and $B\leq_{\mu^*e} M$.
(2) Let $A_1\leq_{\mu^*e} B_1\leq M$ and $A_2\leq_{\mu^*e} B_2\leq M$, then $A_1\cap A_2\leq_{\mu^*e} B_1\cap B_2$.
(3) If $f : M_1\rightarrow M_2$ is an R- homomorphism and $A\leq_{\mu^*e} M_2$, then $f^{-1}(A)\leq_{\mu^*e} M_1$.
(4) Let $\{A_\alpha\}\ \alpha\in\Lambda$ be an independent family of submodules of M and $A_\alpha\leq_{\mu^*e} B_\alpha$, $\forall\alpha\in\Lambda$, then $\bigoplus_{\alpha\in\Lambda} A_\alpha\leq_{\mu^*e} \bigoplus_{\alpha\in\Lambda} B_\alpha$.

Proof. (1) Suppose that $A\leq_{\mu^*e} M$ and let L be a nonzero singular submodule of B. Since $A\leq_{\mu^*e} M$, then $A\cap L\neq 0$. Hence $A\leq_{\mu^*e} B$. Now let K be a nonzero singular submodule of M , then $0\neq A\cap K\leq B\cap K$. Thus $B\leq_{\mu^*e} M$.

Conversely , assume that $A\leq_{\mu^*e} B\leq_{\mu^*e} M$ and let L be a nonzero singular submodule of M , then $B\cap L$ is a nonzero singular submodule of B. But $A\leq_{\mu^*e} B$, therefore $A\cap B\cap L = A\cap L\neq 0$. Thus we get the result.

(2) Assume that $A_1\leq_{\mu^*e} B_1\leq M$ and $A_2\leq_{\mu^*e} B_2\leq M$ and let L be a nonzero singular submodule of $B_1\cap B_2\leq B_1$. Since $A_1\leq_{\mu^*e} B_1$, then $A_1\cap L\neq 0$ and hence it is a nonzero singular submodule of B_2. But $A_2\leq_{\mu^*e} B_2$, therefore $A_1\cap A_2\cap L\neq 0$. Thus $A_1\cap A_2\leq_{\mu^*e} B_1\cap B_2$.

(3) Let $f : M_1\rightarrow M_2$ be an R- homomorphism and let $A\leq_{\mu^*e} M_2$. To show that $f^{-1}(A)\leq_{\mu^*e} M_1$, let $0\neq x\in M_1$ with Rx is singular submodule of M_1, then $f(Rx)$ is a singular submodule of M_2. Consider the following two cases.

(a) if $x\in f^{-1}(A)$, we are done.

(b) if $x\notin f^{-1}(A)$, $0\neq f(x)\in M_2$. Since $A\leq_{\mu^*e} M_2$, then there is $r\in R$ such that $0\neq r f(x)\in A$, hence $0\neq rx\in f^{-1}(A)$. Thus $f^{-1}(A)\leq_{\mu^*e} M_1$.

(4) We use the induction on the number of elements of Λ. Suppose that the family has only two elements. i.e. , $\{A_1 , A_2\}$ is independent family in M, $A_1\leq_{\mu^*e} B_1$ and $A_2\leq_{\mu^*e} B_2$. Let $\pi_1 : B_1\oplus B_2\rightarrow B_1$ and $\pi_2 : B_1\oplus B_2\rightarrow B_2$ be the projection maps. Since $A_1\leq_{\mu^*e} B_1$ and $A_2\leq_{\mu^*e} B_2$, then $\pi_1^{-1}(A_1) = A_1\oplus B_2\leq_{\mu^*e} B_1\oplus B_2$ and $\pi_2^{-1}(A_2) = B_1\oplus A_2\leq_{\mu^*e} B_1\oplus B_2$, by(3) and hence $A_1\oplus A_2 = (A_1\oplus B_2)\cap(B_1\oplus A_2)\leq_{\mu^*e} B_1\oplus B_2$, by (2).

Now, assume that the result is true for the case when the index set with $n-1$ elements. Now let $\{A_1,A_2,\ldots,A_n\}$ be an independent family and assume that $A_i\leq_{\mu^*e} B_i$, $\forall\ i = 1, 2,\ldots,n$. By the previous case we have $\bigoplus_{i=1}^{n-1} A_i\leq_{\mu^*e}\bigoplus_{i=1}^{n-1} B_i$ and

$A_n \leq_{\mu^*e} B_n$, hence we get $\bigoplus_{i=1}^{n} A_i \leq_{\mu^*e} \bigoplus_{i=1}^{n} B_i$. Finally, let $\{A_\alpha\}$ $\alpha \in \Lambda$ be an independent family of submodules of M and $A_\alpha \leq_{\mu^*e} B_\alpha$, $\forall \alpha \in \Lambda$. Let N be a nonzero singular submodule of $\bigoplus_{\alpha\in\Lambda} B_\alpha$ and let x be a nonzero element in N. So $x = b_1+b_2+\ldots+b_n$, where $b_i \in B_{\alpha_i}$, $\forall\ i = 1,2,\ldots,n$. Hence $N \cap (A_{\alpha 1}+ A_{\alpha 2}+\ldots+ A_{\alpha n}) \neq 0$ which implies that $N \cap \bigoplus_{\alpha\in\Lambda} A_\alpha \neq 0$. Thus $\bigoplus_{\alpha\in\Lambda} A_\alpha \leq_{\mu^*e} \bigoplus_{\alpha\in\Lambda} B_\alpha$. □

Notes. (1) Note that $\{B_\alpha\}_{\alpha\in\Lambda}$ in proposition (2.8-4) need not be an independent family. Example: Let M be the Z- module $Z \oplus Z_2$ and let $A_1= 0 \oplus Z_2$, $B_1= Z \oplus Z_2$, $A_2 = B_2 = Z \oplus \bar{0}$. One can easily show that $A_1 \leq_{\mu^*e} B_1$ and $A_2 \leq_{\mu^*e} B_2$ and $A_1 \cap A_2 = \{0\}$ but $B_1 \cap B_2 = Z \oplus \bar{0}$. Hence $\{B_1,B_2\}$ is not independent family.

(2) Let A_1, A_2, B_1 and B_2 be submodules of an R- module M. If $A_1 \leq_{\mu^*e} B_1$ and $A_2 \leq_{\mu^*e} B_2$, then it is not necessary that $(A_1+A_2) \leq_{\mu^*e} (B_1+B_2)$ as the following example shows:

Consider the Z- module $Z \oplus Z_2$. Let $A_1 = A_2 = Z(\bar{2}, \bar{0})$ and $B_1 = Z(\bar{1}, \bar{0})$, $B_2 = Z(\bar{1}, \bar{1})$. One can easily show that $A_1 \leq_{\mu^*e} B_1$ and $A_2 \leq_{\mu^*e} B_2$. But (A_1+B_1) is not μ*-essential in (B_1+B_2) , where there exists a nonzero singular submodule $K = \{\bar{0}\} \oplus Z_2$ of (B_1+B_2) such that $(A_1+A_2) \cap K = \{(\bar{0}, \bar{0})\}$.

Recall that a submodule A of an R- module M is called a **closed submodule** of M if A has no proper essential extension. See [1].

Now, we define the μ*- closed submodules and introduce the basic properties of these submodules.

Definition (2.9): Let A be a submodule of an R- module M, we say that A is **μ*-closed in *M*** (briefly $A \leq_{\mu^*c} M$) if A has no proper μ*- essential extension in M.

The following proposition ensure the existences of μ*- closed submodules.

Proposition (2.10): Let M be an R- module . Then every submodule is μ*- essential in μ*- closed submodule of M.

Proof: Let A be a submodule of M. Consider the collection $\Gamma = \{K: K \leq M: A \leq_{\mu^*e} K\}$. It is clear that Γ is nonemplty set . Let $\{C_\alpha\}$ $\alpha \in \Lambda$ be a chain in Γ. To show that $A \leq_{\mu^*e} \bigcup_{\alpha\in\Lambda} C_\alpha$, let $0 \neq x \in \bigcup_{\alpha\in\Lambda} C_\alpha$ with Rx is singular submodule of $\bigcup_{\alpha\in\Lambda} C_\alpha$, then there is $\alpha_\circ \in \Lambda$ such that $0 \neq x \in C_{\alpha_\circ}$.But $A \leq_{\mu^*e} C_\alpha$, $\forall \alpha \in \Lambda$, therefore there exists $r \in R$ such that $0 \neq rx \in A$, hence $A \leq_{\mu^*e} \bigcup_{\alpha\in\Lambda} C_\alpha$ which means that $\bigcup_{\alpha\in\Lambda} C_\alpha \in \Gamma$. By Zorn's lemma Γ has a maximal element say H. To show that H is μ*- closed in M , let B be a submodule of M such that $H \leq_{\mu^*e} B$, then $A \leq_{\mu^*e} H \leq_{\mu^*e} B$ and hence $A \leq_{\mu^*e} B$, by Prop. (2.8) . But H is maximal element in Γ. Thus $H = B$. □

Remarks and Examples (2.11).

(1) Every μ*- closed submodule of an R- module M is closed in M. The converse is not true in general. For example , Consider Z_6 as Z_6- module $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are closed in Z_6 but not μ*- closed in Z_6.

(2) Consider Z_6 as Z- module , $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are μ*-closed submodules of Z_6.

(3) In Z_4 as Z- module , $\{\bar{0}, \bar{2}\}$ is not μ*- closed in Z_4.

(4) Let M be a singular R- module. Then A is closed in M if and only if A is μ*- closed in M.

(5) Let M be a torsion module over a commutative integral domain R and A be a submodule of M. Then $A \leq_{\mu^*c} M$ if and only if $A \leq_c M$.

(6) Let M be a prime R- module with $Z(M) \neq 0$ and A be a submodule of M. Then $A \leq_{\mu^*c} M$ if and only if $A \leq_c M$.

(7) It is well known that every direct summand of an R- module M is closed in M. But in case μ*-closed there is no relationship with direct summands. For example , Z_6 as Z_6- module , the nontrivial direct summands of Z_6 are $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ which are not μ*- closed in Z_6.

(8) If a submodule A of an R- module M is μ*- closed and μ*- essential in M , then $A = M$.

(9) The intersection of μ*- closed submodules of M need not be μ*- closed in M. For example , consider $M = Z \oplus Z_2$ as Z- module , let $A = Z \oplus \bar{0}$, $B = Z(1, \bar{1})$. Since $0 \oplus Z_2$ is the only singular submodule of M and has zero intersection with A , then $A \leq_{\mu^*c} M$. Similarly $B \leq_{\mu^*c} M$, but $A \cap B = 2Z \oplus \bar{0}$ which is not μ*- closed in M.

Next, we give the basic properties of μ*-closed submodules.

Proposition (2.12): Let M be an R- module. If $A \leq_{\mu^*c} M$, then $\frac{B}{A} \leq_{\mu^*e} \frac{M}{A}$, whenever $B \leq_{\mu^*e} M$ with $A \leq B$.

Proof. Suppose that $A \leq B \leq_{\mu^*e} M$ and let $\frac{L}{A}$ be a singular submodule of $\frac{M}{A}$ such that $\frac{L}{A} \cap \frac{B}{A} = A$, then $L \cap B =$

A. Since $B \leq_{\mu^*e} M$, then $A \leq_{\mu^*e} L$, by Prop. (2.8-2). But A is µ*- closed in M , therefore $A = L$. Thus $\frac{B}{A} \leq_{\mu^*e} \frac{M}{A}$. □

Proposition (2.13): Let $f : M \to M'$ be an epimorphism and let A be a submodule of M such that $Kerf \leq A$. If A is µ*- closed in M , then $f(A)$ is µ*- closed in M'.

Proof. Let K' be a submodule of M' such that $f(A) \leq_{\mu^*e} K'$, then $f^{-1}(f(A)) \leq_{\mu^*e} f^{-1}(K')$, by Prop. (2.8). One can easily show that $f^{-1}(f(A)) = A$, hence $A \leq_{\mu^*e} f^{-1}(K')$. But A is µ*- closed in M , therefore $A = f^{-1}(K')$, and hence $f(A) = K'$. Thus $f(A)$ is µ*- closed in M'. □

One can easily prove the following corollaries.

Corollary (2.14): µ*- closed submodule is closed under isomorphism.

Corollary (2.15): Let A and B be submodules of an R- module M with $A \leq B$. If B is µ*- closed in M, then $\frac{B}{A}$ is µ*- closed in $\frac{M}{A}$.

Proposition (2.16): Let M be an R- module and let A , B be submodules of M with $A \leq B \leq M$. If A is µ*- closed in M , then A is µ*- closed in B.

Proof: Suppose that $A \leq_{\mu^*e} L \leq B \leq M$. But A is µ*- closed in M , therefore $A = L$. Thus A is µ*- closed in B. □

It is easy to prove the following corollary.

Corollary (2.17): Let A and B be submodules of an R- module M if $A \cap B$ is µ*- closed in M , then $A \cap B$ is µ*- closed in A and B.

We cannot prove the transitive property for µ*- closed submodules. However under certain condition we can prove this property as we see in the following result.

Recall that an R- module M is called **chained module** if for each submodules A and B of M either $A \leq B$ or $B \leq A$, see [7].

Proposition (2.18): Let M be chained R- module and let A and B be submodules of M such that $A \leq B \leq M$. If $A \leq_{\mu^*c} B \leq_{\mu^*c} M$, then $A \leq_{\mu^*c} M$.

Proof. Let K be a submodule of M such that $A \leq_{\mu^*e} K \leq M$. By our assumption we have two cases: If $K \leq B$. Since A is µ*- closed in B , then $A = K$, hence $A \leq_{\mu^*c} M$. If $B \leq K$, since $A \leq_{\mu^*e} K$, so $B \leq_{\mu^*e} K$, by Prop. (2.8). But $B \leq_{\mu^*c} M$, therefore $B = K$, hence $A \leq_{\mu^*e} B$. But $A \leq_{\mu^*c} B$, therefore $A = B = K$. Thus A is µ*- closed in M. □

The following proposition shows that the direct sum of µ*-closed submodules is again µ*- closed .

Proposition (2.19): Let M_1, M_2 be two R- modules. If $A_1 \leq_{\mu^*c} M_1$ and $A_2 \leq_{\mu^*c} M_2$, then $A_1 \oplus A_2 \leq_{\mu^*c} M_1 \oplus M_2$.

Proof: Assume that $A_1 \oplus A_2 \leq_{\mu^*e} B_1 \oplus B_2$, $B_1 \leq M_1$ and $B_2 \leq M_2$, let $i_1: M_1 \to M_1 \oplus M_2$ and $i_2: M_2 \to M_1 \oplus M_2$ be the inclusion maps. Since $A_1 \oplus A_2 \leq_{\mu^*e} B_1 \oplus B_2$, then $i_1^{-1}(A_1 \oplus A_2) \leq_{\mu^*e} i_1^{-1}(B_1 \oplus B_2)$. Note that $i_1^{-1}(A_1 \oplus A_2) = \{x \in M_1: i_1(x) \in (A_1 \oplus A_2)\} = \{x \in M_1: (x,0) \in (A_1 \oplus A_2)\} = A_1 \leq_{\mu^*e} i_1^{-1}(B_1 \oplus B_2) = B_1$. Similarly , $A_2 \leq_{\mu^*e} B_2$. But $A_1 \leq_{\mu^*c} M_1$ and $A_2 \leq_{\mu^*c} M_2$, therefore $A_1 = B_1$ and $A_2 = B_2$. Thus $A_1 \oplus A_2 \leq_{\mu^*c} M_1 \oplus M_2$. □

An R- module M is called **uniform** module if every nonzero submodule of M is essential in M , see [1].

Now , we introduce µ*- uniform modules as a generalization of uniform modules which is a dual of µ-hollow modules.

Definition (2.20): An R- module M is called **µ*- uniform** if every nonzero submodule of M is µ*- essential in M.

Remarks and Examples (2.21):

(1) Every nonsingular module is µ*- uniform. The converse is not true in general , for example , Z_4 as Z-module.
(2) Every torsion free module over a commutative integral domain is µ*- uniform.
(3) Clearly that every uniform module is µ*- uniform , hence Q as Z- module and Z- as Z- module are µ*- uniform modules.
(4) The converse of (3) is not true in general. For example , Z_6 as Z_6- module.
(5) Z_6 as Z- module is not µ*- module.
(6) Let M be a singular R- module. Then M is uniform if and only if M is µ*- uniform.
(7) Let M be a torsion module over a commutative integral domain R. Then M is uniform if and only if M is µ*- uniform.
(8) Let M be a prime R- module with $Z(M) \neq 0$. Then M is uniform if and only if M is µ*- uniform.

The following theorem gives a characterization of µ*- uniform modules. Compare with [3, theorem (3.7)].

Proposition (2.22): Let M be an R- module. Then M is µ*- uniform if and only if every nonzero singular submodule of M is essential in M.

Proof: ($\Rightarrow$) Assume that M is μ*- uniform and let A be a nonzero singular submodule of M. Assume that there exists a nonzero submodule B of M such that $A \cap B = 0$. Since M is μ*- uniform , then $B \leq_{\mu^*e} M$ and we have A is nonzero singular submodule of M , then $A \cap B \neq 0$, which is a contradiction.

($\Leftarrow$) To show that M is μ*- uniform , let A be a nonzero submodule of M and assume that A is not μ*- essential in M , that is there exists a nonzero singular submodule B of M such that $A \cap B = 0$. By our assumption $B \leq_e M$, then $A = 0$, which is a contradiction. □

Compare the following Prop. with [3, Prop. (3.8)]

Proposition (2.23): A nonzero monomorphic image of μ*- uniform is μ*- uniform.

Proof: Let $f : M \rightarrow M'$ be an R- monomorphism and assume that M is μ*- uniform , we have to show that M' is μ*- uniform , let A be a nonzero submodule of M', then $f(A) \neq 0$, if $f(A) = 0$, then $A \leq Kerf = 0$ which is a contradiction. Since M' is μ*- uniform , then $f(A) \leq_{\mu^*e} M'$ and hence $A \leq_{\mu^*e} M$. □

Corollary (2.24): A submodule of μ*- uniform is again μ*- uniform.

Note. A quotient of μ*- uniform need not be μ*- uniform. For example , Z as Z- module is μ*- uniform but $\frac{Z}{6Z} \simeq Z_6$ which is not μ*- uniform.

The following proposition gives a condition under which a quotient of μ*- uniform is μ*- uniform.

Proposition (2.25): Let M be a μ*- uniform and let A be a μ*- closed submodule of M , the

n $\frac{M}{A}$ is μ*- uniform.

Proof: Let $\frac{L}{A}$ be a nonzero submodule of $\frac{M}{A}$, hence L is nonzero submodule of M. But M is μ*- uniform , therefore $L \leq_{\mu^*e} M$. Since A is μ*- closed in M , then $\frac{L}{A} \leq_{\mu^*e} \frac{M}{A}$, by Prop. (2.12). Thus $\frac{M}{A}$ is μ*- uniform. □

A direct sum of μ*- uniform modules need not be μ*- uniform. For example , let $M = Z_8 \oplus Z_2$ as Z- module, clearly that Z_8 and Z_2 are μ*- uniform Z- modules but M is not μ*- uniform , where there exists a singular submodule $A = <(\bar{0}, \bar{1})>$ which is not essential in M since there is $B = <(\bar{2}, \bar{0})>$ such that $A \cap B = 0$.

Now , we give certain conditions under which a direct sum of μ*- uniform modules is μ*- uniform.

Let M be an R- module. Recall that a submodule A of M is called a **fully invariant** if $g(A) \leq A$, for every $g \in End(M)$ and M is called **duo module** if every submodule of M is fully invariant. See [8].

Proposition (2.26): Let $M = M_1 \oplus M_2$ be a duo module. If M_1 and M_2 are μ*- uniform modules , then M is μ*- uniform. Provided that $A \cap M_i \neq 0$, $\forall\ i = 1,2$.

Proof: Let A be a nonzero submodule of M. Since M is duo module , then A is fully invariant and hence $A = (A \cap M_1) \oplus (A \cap M_2)$. Since each of $(A \cap M_1)$ and $(A \cap M_2)$ is a nonzero submodule of M_1 and M_2 respectively , it follows that $(A \cap M_1) \leq_{\mu^*e} M_1$ and $(A \cap M_2) \leq_{\mu^*e} M_2$. Then $A \leq_{\mu^*e} M$, by Prop. (2.8). □

Recall that an R- module M is called **distributive** if for all A , B and $C \leq M$, $A \cap (B+C) = (A \cap B)+(A \cap C)$. See [9].

In similar argument one can easily prove the following proposition.

Proposition (2.27): Let $M = M_1 \oplus M_2$ be a distributive module. If M_1 and M_2 are μ*- uniform modules , then M is μ*- uniform. Provided that $A \cap M_i \neq 0$, $\forall\ i = 1,2$.

3. μ*-Extending modules.

In this section , we introduce the concept of μ*- extending modules as a generalization of extending modules. We generalize some properties of extending modules to μ*- extending modules and discuss when the submodule of μ*- extending module is μ*- extending module.

Definition (3.1): An R- module M is called **μ*- extending module** if every submodule of M is μ*- essential in a direct summand. Clearly that every μ*- uniform module is μ*- extending. The converse is not true in general. For example , Z_6 as Z- module.

Remarks and Examples (3.2).

(1) Every extending module is μ*- extending , hence Z as Z- module is μ*- extending. The converse is not true in general . For example , let $R = Z[x]$ be a polynomial ring of integers Z and let $M = Z[x] \oplus Z[x]$. Since M is nonsingular , then it is μ*- uniform and hence it is μ*- extending , but M is not extending , by [2 , P.109].

(2) Let M be a singular R- module. Then M is μ*- extending if and only if M is extending.
(3) Let M be a torsion module over a commutative integral domain. Then M is μ*- extending if and only if M is extending.
(4) Let M be a prime R- module with $Z(M) \neq 0$. Then M is μ*- extending if and only if M is extending.
(5) For any prime number p , the Z- module $M = Z_p \oplus Z_{p2}$ is μ*- extending.
(6) For any prime number p , the Z- module $M = Z_p \oplus Z_{p3}$ is not μ*- extending.

The following proposition gives a condition under which the μ*- extending module and μ*- uniform module are equivalent.

Proposition (3.3): Let M be an indecomposable module. Then the following statements are equivalent.

(1) M is μ*- uniform.
(2) M is μ*- extending.
(3) Every cyclic submodule of M is μ*- essential in a direct summand of M.

Proof: $(1)\Rightarrow(2)\Rightarrow(3)$ It is clear.

$(3)\Rightarrow(1)$ Assume that every cyclic submodule of M is μ*- essential in a direct summand of M and let A be a nonzero submodule of M , let $x\in A$, hence Rx is μ*- essential in a direct summand D of M. But M is indecomposable, therefore $D = M$. Since $Rx \leq A \leq M$, then $A \leq_{\mu^*e} M$. Thus M is μ*- uniform. □

Now , we give various conditions under which a submodule of a μ*- extending module is μ*- extending.

Proposition (3.4): Let M be a μ*- extending R- module and let A be a submodule of M such that the intersection of A with any direct summand of M is a direct summand of A, then A is a μ*- extending module.

Proof: Let $X \leq A \leq M$. Since M is μ*- extending , then there exists a direct summand D of M such that $X \leq_{\mu^*e} D$. By our assumption $A \cap D$ is a direct summand of A. Hence $X = (X \cap A) \leq_{\mu^*e} (A \cap D)$, by Prop. (2.8). Thus A is μ*- extending.

□

Let M be an R- module. Recall that a submodule A of M is called a **fully invariant** if $g(A) \leq A$, for every $g \in \mathrm{End}(M)$ and M is called **duo module** if every submodule of M is fully invariant. See [8].

Proposition (3.5): Every fully invariant submodule of μ*- extending module is μ*- extending.

Proof. Let M be a μ*- extending module and let A be a fully invariant submodule of M. Let X be a submodule of A. Since M is μ*- extending , then there exists a direct summand D of M such that $X \leq_{\mu^*e} D$. Let $M = D \oplus D'$, where $D' \leq M$. Now consider the projection map $p: M \longrightarrow D$,then $(1-p): M \longrightarrow D'$. Claim that $A=(A \cap p(M)) \oplus ((I-p)(M) \cap A)$. To show that , let $x \in A$, then $x = a+b$, $a \in D$ and $b \in D'$. Now $P(x) = p(a+b) = a$ and $(1-p)(x) = b$. But A is fully invariant , therefore $p(x) = a \in p(M) \cap A$ and $(1-p)(x) = b \in (1-p)(M) \cap A$. Thus $A=(A \cap p(M)) \oplus ((I-p)(M) \cap A) = (A \cap D) \oplus (A \cap D')$. Since $X \leq_{\mu^*e} D$, then $X=(X \cap A) \leq_{\mu^*e} (A \cap D)$.Thus A is μ*-extending , by Prop.(2.8). □

Corollary (3.6): Let M be a duo μ*- extending module , then every submodule of M is μ*- extending.

The next proposition gives another condition under which the submodule of μ*- extending module is a μ*- extending.

Recall that an R- module M is called **distributive** if for all A , B and $C \leq M$, $A \cap (B+C) = (A \cap B)+(A \cap C)$. See [9].

Proposition (3.7): Let M be a distributive μ*- extending R- module, then every submodule of M is μ*- extending.

Proof: Let A be a submodule of M and let X be a submodule of A. Since M is μ*- extending , then there exists a direct summand D of M such that $X \leq_{\mu^*e} D$, let $M=D \oplus D'$, where $D' \leq M$. But M is distributive, therefore $A=(A \cap D) \oplus (A \cap D')$, then $(A \cap D)$ is a direct summand of A and $X \leq_{\mu^*e} (A \cap D)$. Thus A is μ*-extending. □

Let M be an R- module. Recall that a proper submodule A of M is called a **maximal submodule** if whenever $A \subset B \leq M$, then $B = M$. Equivalently , A is maximal submodule if $M = Rx+A$, $\forall x \notin A$, see [10].

Proposition (3.8): Let M be a μ*- extending module which contains maximal submodules. Then for any maximal submodule A of M , either $A \leq_{\mu^*e} M$ or $M = A \oplus B$, for some simple submodule B of M.

Proof: Let A be a maximal submodule of M and suppose that A is not μ*- essential submodule of M , then there is a nonzero singular submodule B of M such that $A \cap B = 0$, let $x \in B$ and $x \notin A$. Since A is maximal submodule of M , then $M = A + Rx \leq A+B$, hence $M = A \oplus B$. Since $B \cong \frac{M}{A}$, so B is simple. □

A module M is called **local module** if it has a largest submodule , i.e, a proper submodule which contains all other proper submodules. For a local module M, Rad(M) , the Jacobson radical of M is small in M , see [11].

Corollary (3.9): Let M be a local μ*- extending module , then $Rad(M)\leq_{\mu^*e} M$.

Proof: Since M is local module , then $Rad(M)<<M$, hence $Rad(M)$ can not be a direct summand of M. Thus $Rad(M)\leq_{\mu^*e} M$, by Prop. (3.8). □

4. Characterizations of μ*-extending modules.

In this section , we give various characterizations of μ*- extending modules. Also, we give some conditions under which the direct sum of μ*- extending modules is μ*- extending module.

Theorem (4.1): Let M be an R- module. Then M is μ*- extending module if and only if every μ*- closed submodule of M is a direct summand.

Proof: ($\Rightarrow$) Suppose that M is μ*- extending and let A be a μ*- closed in M , then there is a direct summand D of M such that $A\leq_{\mu^*e} D$. But A is μ*- closed in M , therefore $A = D$.

($\Leftarrow$) To show that M is μ*- extending , let A be a submodule of M , then there is a μ*- closed submodule B of M such that $A\leq_{\mu^*e} B$, by Prop. (2.10). By our assumption , B is a direct summand of M. Thus M is μ*- extending module. □

Theorem (4.2): Let M be an R- module. Then the following statements are equivalent.

(1) M is μ*- extending module.

(2) For every submodule A of M , there is a decomposition $M = D\oplus D'$, such that $A \leq D$ and $D'+A \leq_{\mu^*e} M$.

(3) For every submodule A of M , there is a decomposition $\frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A}$ such that D is a direct summand of M and $K\leq_{\mu^*e} M$.

Proof: (1)$\Rightarrow$(2) Let M be a μ*- extending and let A be a submodule of M , there is a direct summand D of M such that $A \leq_{\mu^*e} D$, then $M = D\oplus D'$, $D' \leq M$. Since $\{A , D'\}$ is an independent family , then $A+D' \leq_{\mu^*e} M$, by Prop. (2.8).

(2)$\Rightarrow$(3) Let A be a submodule of M. By (2) , there is a decomposition $M = D\oplus D'$, such that $A \leq D$ and $D'+A \leq_{\mu^*e} M$. Claim that $\frac{M}{A}=\frac{D}{A}\oplus\frac{D'+A}{A}$. Since $M = D\oplus D'$, then $\frac{M}{A}=\frac{D+D'}{A}=\frac{D}{A}+\frac{D'+A}{A}$ and $\frac{D}{A}\cap\frac{D'+A}{A}=\frac{D\cap(D'+A)}{A}=\frac{A+(D\cap D')}{A}=A$, hence $\frac{M}{A}=\frac{D}{A}\oplus\frac{D'+A}{A}$. Take $K = D'+A$, so we get the result.

(3)$\Rightarrow$(1) To show that M is μ*- extending , let A be a submodule of M. By (3) , there is a decomposition $\frac{M}{A} = \frac{D}{A}\oplus\frac{K}{A}$ such that D is a direct summand of M and $K\leq_{\mu^*e} M$. It is enough to show that $A \leq_{\mu^*e} D$. Let $i :D\rightarrow M$ be the injection map. Since $K\leq_{\mu^*e} M$, then $i^{-1}(K) \leq_{\mu^*e} i^{-1}(M)$, that is $D\cap K\leq_{\mu^*e} D$. One can easily show that $D\cap K = A$, so M is μ*- extending module. □

Proposition (4.3): Let M be an R- module. Then M is μ*- extending module if and only if for each μ*- closed submodule A of M , there is a complement B of A in M such that every homomorphism $f : A\oplus B\rightarrow M$ can be lifted to a homomorphism $g : M\rightarrow M$.

Proof: This is a direct consequence of [12 , Lemma 2]. □

Proposition (4.4): Let M be an R- module. Then M is μ*- extending module if and only if for every submodule A of M , there exists an idempotent $f \in$ End (M) such that $A \leq_{\mu^*e} f(M)$.

Proof: Clear.

The following proposition gives another characterization of μ*- extending module.

Proposition (4.5): Let M be an R- module , then M is μ*- extending module if and only if for each direct summand A of the injective hull $E(M)$ of M , there exists a direct summand D of M such that $(A\cap M)\leq_{\mu^*e} D$.

Proof: Let A be a submodule of M and let B be a complement of A , then $A\oplus B \leq_e M$,by [1, Prop. (1.3)]. Since $M\leq_e E(M)$, then $A\oplus B \leq_e E(M)$. Thus $E(A)\oplus E(B) = E(A\oplus B) = E(M)$. By our assumption , there exists a direct summand D of M such that $E(A)\cap M \leq_{\mu^*e} D$. But $A \leq_e E(A)$, therefore $A\cap M\leq_{\mu^*e} E(A)\cap M \leq_{\mu^*e} D$, hence $A \leq_{\mu^*e} D$. Thus M is μ*- extending. The proof of the converse is clear. □

The following proposition shows that the direct summand of μ*- extending module is μ*- extending.

Proposition (4.6): A direct summand of μ*- extending module is μ*- extending.

Proof: Let $M = A \oplus B$ be a μ*- extending module. To show that A is a μ*- extending , let X be a μ*- closed submodule of A , then $X \oplus B$ is a μ*- closed submodule of M , by Prop. (2.19). Hence $X \oplus B$ is a direct summand of M , then $M = X \oplus B \oplus Y$, $Y \leq M$, that is X is a direct summand of M. But $X \leq A$, therefore X is a direct summand of A. Thus A is μ*- extending module. □

The following proposition gives a condition under which a quotient of μ*- extending module is a μ*- extending.

Proposition (4.7): Let M be a μ*- extending module and let A be a μ*- closed submodule of M , then $\frac{M}{A}$ is μ*- extending module.

Proof: Let M be a μ*- extending module and let A be a μ*- closed submodule of M , then A is a direct summand of M , let $M = A \oplus A'$, for some submodule A' of M , hence $\frac{M}{A} \cong A'$ is a μ*- extending module , by Prop. (3.6). □

Corollary (4.8): Assume that $f : M \rightarrow M'$ is an R- homomorphism and let $Kerf$ be a μ*- closed submodule of M , then $f(M)$ is μ*- extending.

Proof: Let $f : M \rightarrow M'$ be an R- homomorphism and let $Kerf$ be a μ*- closed submodule of M , then $\frac{M}{Kerf} \cong f(M)$ is μ*- extending module. □

The direct sum of μ*- extending modules need not be μ*- extending , for example , let $M = Z_8 \oplus Z_2$ as Z- module, clearly that Z_8 and Z_2 are μ*- extending Z- module but M is not μ*- extending.

Now , we give sufficient conditions under which the direct sum of μ*-extending modules is a μ*-extending.

Proposition (4.9): Let $M=M_1 \oplus M_2$ be a distributive module if M_1 and M_2 are μ*-extending , then M is μ*- extending.

Proof: Let $M = M_1 \oplus M_2$ be a distributive module , M_1 and M_2 are μ*-extending and let $A \leq M$. Since M is distributive, then $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since M_1 , M_2 are μ*-extending , then there exists a direct summand D_1 of M_1 and direct summand D_2 of M_2 such that $(A \cap M_1) \leq_{\mu^* e} D_1$ and $(A \cap M_2) \leq_{\mu^* e} D_2$. Hence $A = (A \cap M_1) \oplus (A \cap M_2)) \leq_{\mu^* e} (D_1 \oplus D_2)$ and $D_1 \oplus D_2$ is a direct summand of M , by Prop. (2.8). Thus M is μ*-extending. □

Proposition (4.10): Let $M = \bigoplus_{i \in I} M_i$ be an R-module ,where M_i is a submodule of M ,$\forall\ i \in I$. If M_i is μ*-extending , for each $i \in I$ and every μ*- closed submodule of M is fully invariant , then M is μ*-extending.

Proof: Let A be a μ*- closed submodule of M and $\pi_i : M \longrightarrow M_i$ be the natural projection on M_i , for each $i \in I$. Let $x \in A$, then $x = \sum x_i$, $i \in I$, $x_i \in M_i$, $\pi_i(x) = x_i$. By our assumption, A is fully invariant and hence $\pi_i(A) \leq A \cap M_i$. So, $\pi_i(x) = x_i \in A \cap M_i$ and hence $x \in \bigoplus_{i \in I} (A \cap M_i)$. Thus $A \leq \bigoplus_{i \in I} (A \cap M_i)$. But $\bigoplus_{i \in I} (A \cap M_i) \leq A$, therefore $A = \bigoplus_{i \in I} (A \cap M_i)$, $\forall\ i \in I$ .Since $A \cap M_i \leq M_i$ and M_i is μ*- extending , then there exists direct summands D_i of M_i such that $(A \cap M_i) \leq_{\mu^* e} D_i$. By Prop. (2.8) $A = (\bigoplus_{i \in I} (A \cap M_i)) \leq_{\mu^* e} (\bigoplus_{i \in I} D_i)$, for each $i \in I$. Thus M is μ*-extending. □

Proposition (4.11) Let M_1 and M_2 be μ*-extending modules such that $annM_1 + annM_2 = R$, then $M_1 \oplus M_2$ is μ*- extending.

Proof: Let A be a submodule of $M_1 \oplus M_2$. Since $annM_1 + annM_2 = R$, then by the same way of the proof of [13 ,Prop.4.2,CH.1] $A = B \oplus C$, where B is a submodule of M_1 and C is a submodule of M_2. Since M_1 and M_2 are μ*- extending , then there exists direct summands D_1 of M_1 and D_2 of M_2 such that $B \leq_{\mu^* e} D_1$ and $C \leq_{\mu^* e} D_2$, hence $A = (B \oplus C) \leq_{\mu^* e} (D_1 \oplus D_2)$, by Prop. (2.8). Thus M is μ*-extending. □

Proposition (4.12): Let $M = M_1 \oplus M_2$ be an R- module with M_1 being μ*- extending and M_2 is semisimple. Suppose that for any submodule A of M with $A \cap M_1$ is a direct summand of A. Then M is μ*- extending.

Proof: Let A be a submodule of M. Then it is easy to see that $A+M_1 = M_1 \oplus [(A+M_1) \cap M_2]$. Since M_2 is semisimple , then $(A+M_1) \cap M_2$ is a direct summand of M_2 and therefore $A+M_1$ is a direct summand of M. By our assumption $A = (A \cap M_1) \oplus A'$, for some submodule A' of M. Since M_1 is μ*- extending , then there is a direct summand D of M_1 such that $A \cap M_1 \leq_{\mu^* e} D$. Hence $A = (A \cap M_1) \oplus A' \leq_{\mu^* e} D \oplus A'$. Since $D \oplus A' \leq_{\oplus} A+M_1 \leq_{\oplus} M$, then $D \oplus A'$ is a direct summand of M. Thus M is μ*- extending. □

Proposition (4.13): Let $M = M_1 \oplus M_2$ with M_1 being μ*-extending and M_2 injective. Suppose that for any submodule A of M, we have $A \cap M_2$ is a direct summand of A, then M is μ*- extending.

Proof: Let A be a submodule of M. By hypothesis, there is a submodule A' of A such that $A = (A \cap M_2) \oplus A'$. Note that $A' \cap M_2 = 0$ and hence $\frac{M_2 + A'}{A'} \cong M_2$ is an injective module, so there is a submodule M' of M such that $\frac{M}{A'} = \frac{M_2 + A'}{A'} \oplus \frac{M'}{A'}$. Thus it is easy to see that $M = M_2 \oplus M'$ and that $M' \cong \frac{M}{M_2} \cong M_1$. Since M_1 is μ*- extending, then M' is μ*- extending, there is a direct summand K of M' such that $M = K \oplus K'$ and $A' \leq_{\mu^*e} K$. Since $A \cap M_2$ is a submodule of M_2 and M_2 is an injective module, then there is a direct summand D of M_2 such that $A \cap M_2 \leq_{\mu^*e} D$. Hence $A = [(A \cap M_2) \oplus A'] \leq_{\mu^*e} D \oplus K$, where $D \oplus K$ is a direct summand of M. Thus M is μ*- extending. □

Proposition (4.14): Let $M = M_1 \oplus M_2$ such that M_1 is μ*-extending and M_2 is injective module. Then M is μ*-extending module if and only if for every submodule A of M such that $A \cap M_2 \neq 0$, there is a direct summand D of M such that $A \leq_{\mu^*e} D$.

Proof: Suppose that for every submodule A of M such that $A \cap M_2 \neq 0$, there is a direct summand D of M such that $A \leq_{\mu^*e} D$. Let A be a submodule of M such that $A \cap M_2 = 0$. Since $\frac{M_2 + A}{A} \cong M_2$ is an injective module, there is a submodule M' of M containing A such that $\frac{M}{A} = \frac{M'}{A} \oplus \frac{(M_2 + A)}{A}$. It is easy to see that $M = M' \oplus M_2$. Since $M' \cong \frac{M}{M_2} \cong M_1$ is μ*- extending, so there is a direct summand K of M', hence K is a direct summand of M, such that $A \leq_{\mu^*e} K$. Thus M is μ*- extending. The proof of the converse is obvious. □

Proposition (4.15): Let R be a PID, then the following statements are equivalent:

1- $\bigoplus_I R$ is μ*-extending, for every index set I.

2- Every projective R- module is μ*-extending.

Proof: (1)$\Rightarrow$(2) Let M be a projective R- module, then by [10, Corollary (4.4.4), p.89], there exists a free R- module F and an epimorphism $f : F \longrightarrow M$. Since F is free, then $F \cong \bigoplus_I R$, for some index set I. Now consider the following short exact sequence:

$$0 \longrightarrow Kerf \xrightarrow{i} \bigoplus_I R \xrightarrow{f} M \longrightarrow 0$$

Where i is the inclusion map. Since M is projective, then the sequence splits. Thus $\bigoplus_I R = Kerf \oplus M$. Since $\bigoplus_I R$ is μ*-extending, then M is μ*- extending, by Prop. (4.6).

(2)$\Rightarrow$(1) Clear. □

By the same argument, we can prove the following:

Proposition(4.16): Let R be a PID, then the following statements are equivalent:

1- $\bigoplus_I$ R is μ*-extending, for every finite index set I.

2- Every finitely generated projective R- module is μ*-extending.

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