Generalized shift operators on ℓ^∞

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Abstract: In the following text we study the compactness of generalized shift operator on $\ell^\infty(\tau)$.

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1. Introduction

One-sided shift $\underset{(x_1,x_2,\cdots)\mapsto(x_2,x_3,\cdots)}{\{1,\ldots,k\}^{\mathbf{N}}\to\{1,\ldots,k\}^{\mathbf{N}}}$ and two-sided shift $\underset{(x_n)_{n\in\mathbf{Z}}\mapsto(x_{n+1})_{n\in\mathbf{Z}}}{\{1,\ldots,k\}^{\mathbf{Z}}\to\{1,\ldots,k\}^{\mathbf{Z}}}$ are amongst most studied maps [6]. Consider arbitrary sets A,Γ with at least two elements and $\varphi:\Gamma\to\Gamma$, we call $\sigma_\varphi: A^\Gamma\to A^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma})=(x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ $((x_\alpha)_{\alpha\in\Gamma}\in A^\Gamma)$ a generalized shift (as a generalization of one-sided and two-sided shifts) which has been introduced for the first time in [2]. Dynamical and non-dynamical properties of generalized shifts have been studied in several texts like [3, 5].

It is well-known that for each (complex) Hilbert space H there exists a unique cardinal number τ such that H and $\ell^2(\tau)=\{(x_\alpha)_{\alpha<\tau}\in\mathbf{C}^\tau:\sum\limits_{\alpha<\tau}|x_\alpha|^2<+\infty\}$ (equipped with inner product $<(x_\alpha)_{\alpha<\tau},(y_\alpha)_{\alpha<\tau}>=\sum\limits_{\alpha<\tau}x_\alpha\overline{y_\alpha}$ and norm $\|(x_\alpha)_{\alpha<\tau}\|=\sqrt{\sum\limits_{\alpha<\tau}|x_\alpha|^2}$), where $\mathbf{C}$ denotes the field of complex numbers. So for $\varphi:\tau\to\tau$ one may consider $\sigma_\varphi|_{\ell^2(\tau)}:\ell^2(\tau)\to\mathbf{C}^\tau$. As it has mentioned in [1], the following statements are equivalent (note that $\sigma_\varphi:\mathbf{C}^\tau\to\mathbf{C}^\tau$ is a linear map):

- $\sigma_\varphi|_{\ell^2(\tau)}(\ell^2(\tau))\subseteq\ell^2(\tau)$,
- $\sigma_\varphi|_{\ell^2(\tau)}(\ell^2(\tau))\subseteq\ell^2(\tau)$ and $\sigma_\varphi|_{\ell^2(\tau)}:\ell^2(\tau)\to\ell^2(\tau)$ is continuous,
- $\varphi:\tau\to\tau$ is bounded, i.e., there exists $K\in\mathbf{N}$ such that for all $\alpha\in\tau$ the set $\varphi^{-1}(\alpha)$ has at most K elements.

In the following text we consider the following Banach space (equipped with norm $\|(x_\alpha)_{\alpha<\tau}\|_\infty=\sup\limits_{\alpha<\tau}|x_\alpha|$):

$$\ell^\infty(\tau)=\{(x_\alpha)_{\alpha<\tau}\in\mathbf{C}^\tau:\sup_{\alpha<\tau}|x_\alpha|<+\infty\}$$

we study $\sigma_\varphi|_{\ell^\infty(\tau)}$.

2. Results on $\sigma_\varphi|_{\ell^\infty(\tau)}$

In this section suppose $\tau\geq 2$ is a cardinal number and $\varphi:\tau\to\tau$ is arbitrary, as our first steps we prove the following theorem.

Theorem 1. We have the following statements:

a. $\sigma_\varphi(\ell^\infty(\tau))\subseteq\ell^\infty(\tau)$,

b. $\sigma_\varphi|_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is continuous and (note that $\|\sigma_\varphi|_{\ell^\infty(\tau)}\|=\sup\{\|\sigma_\varphi(z)\|_\infty:z\in\ell^\infty(\tau),\|z\|_\infty\leq 1\}$):

$$\|\sigma_\varphi|_{\ell^\infty(\tau)}\|=1,$$

c. the following statements are equivalent:

1. $\sigma_\varphi(\ell^\infty(\tau))=\ell^\infty(\tau)$,
2. $\sigma_\varphi(\ell^\infty(\tau))$ is dense in $\ell^\infty(\tau)$,
3. $\varphi:\tau\to\tau$ is one-to-one.

Proof. a, b) Consider $x=(x_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$, then

$$\|\sigma_\varphi(x)\|_\infty=\|\sigma_\varphi((x_\alpha)_{\alpha<\tau})\|_\infty=\|(x_{\varphi(\alpha)})_{\alpha<\tau}\|_\infty$$
$$=\sup_{\alpha<\tau}|x_{\varphi(\alpha)}|\leq\sup_{\alpha<\tau}|x_\alpha|=\|(x_\alpha)_{\alpha<\tau}\|_\infty=\|x\|_\infty$$

and $\|\sigma_\varphi(x)\|_\infty\leq\|x\|_\infty$, hence $\sigma_\varphi(x)\in\ell^\infty(\tau)$, also $\sigma_\varphi|_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is continuous and $\|\sigma_\varphi|_{\ell^\infty(\tau)}\|\leq 1$, on the other hand $(1)_{\alpha<\tau}\in\ell^\infty(\tau)$ and $\|\sigma_\varphi((1)_{\alpha<\tau})\|_\infty=\|(1)_{\alpha<\tau}\|_\infty=1$ which completes the proof of $\|\sigma_\varphi|_{\ell^\infty(\tau)}\|=1$.

c) We complete the proof by showing "$(2)\Rightarrow(3)$" and "$(3)\Rightarrow(1)$".

$(2)\Rightarrow(3)$: Suppose $\varphi:\tau\to\tau$ is not one-t-one, choose $\beta<\theta<\tau$ with $\varphi(\beta)=\varphi(\theta)$. Let $q_\beta=1$ and $q_\alpha=0$ for $\alpha\neq\beta$. Then $U:=\{x\in\ell^\infty(\tau):\|x-(q_\alpha)_{\alpha<\tau}\|_\infty<\frac{1}{2}\}$ is an open neighborhood of $(q_\alpha)_{\alpha<\tau}(\in\ell^\infty(\tau))$, moreover for all $(x_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$ we have

$$\|\sigma_\varphi(x)-(q_\alpha)_{\alpha<\tau}\|_\infty=\|(x_{\varphi(\alpha)})_{\alpha<\tau}-(q_\alpha)_{\alpha<\tau}\|_\infty$$
$$=\sup_{\alpha<\tau}|x_{\varphi(\alpha)}-q_\alpha|\geq\max(|x_{\varphi(\beta)}-q_\beta|,|x_{\varphi(\theta)}-q_\theta|)$$
$$=\max(|x_{\varphi(\beta)}-1|,|x_{\varphi(\theta)}|)\geq\tfrac{1}{2}(|x_{\varphi(\beta)}-1|+|x_{\varphi(\theta)}|)$$
$$\overset{\varphi(\beta)=\varphi(\theta)}{=}\tfrac{1}{2}(|x_{\varphi(\beta)}-1|+|x_{\varphi(\beta)}|)\geq\tfrac{1}{2}|x_{\varphi(\beta)}-1-x_{\varphi(\beta)}|=\tfrac{1}{2}$$

thus $\sigma_\varphi(\ell^\infty(\tau))\cap U$ is empty and $\sigma_\varphi(\ell^\infty(\tau))$ is not dense in $\ell^\infty(\tau)$.

(3)$\Rightarrow$(1): Suppose $\varphi:\tau\to\tau$ is one-to-one and choose $x=(x_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$ define $y=(y_\alpha)_{\alpha<\tau}$ with:

$$y_\alpha:=\begin{cases}x_\beta & \beta<\tau,\alpha=\varphi(\beta),\\ 0 & \text{otherwise}.\end{cases}$$

Then

$$\|y\|_\infty=\sup_{\alpha<\tau}|y_\alpha|=\sup_{\substack{\alpha=\varphi(\beta),\\ \beta<\tau}}|x_\beta|\le\sup_{\alpha<\tau}|x_\alpha|=\|x\|_\infty<+\infty$$

and $y\in\ell^\infty(\tau)$. Moreover $\sigma_\varphi(y)=(y_{\varphi(\alpha)})_{\alpha<\tau}=(x_\alpha)_{\alpha<\tau}$ which completes the proof.

Let's recall that in Banach spaces X,Y we say linear continuous map $T:X\to Y$ is a compact operator if $\overline{\{T(x):\|x\|<1\}}$ is a compact subset of Y [4].

Theorem 2. $\sigma_\varphi\restriction_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is a compact operator if and only if $\varphi(\tau)$ is finite.

Proof. First suppose $\varphi(\tau)$ is infinite. Choose one-to-one sequence $\{\alpha_i\}_{i\ge1}$ in τ such that $\{\varphi(\alpha_i)\}_{i\ge1}$ is a one-to-one sequence too. For each $i\ge1$ let $x_i=(x^i_\alpha)_{\alpha<\tau}\in\ell^\infty(\tau)$ with $x^i_{\alpha_i}=\frac12$ and $x^i_\alpha=0$ for $\alpha\ne\alpha_i$. Then for $i\ne j$ we have $\|\sigma_\varphi(x_i)-\sigma_\varphi(x_j)\|_\infty=\frac12$ and $\{\sigma_\varphi(x_i)\}_{i\ge1}$ does not have any convergent subsequence however $\{x_i\}_{i\ge1}$ is a sequence in $\{x\in\ell^\infty(\tau):\|x\|_\infty<1\}$, so $\sigma_\varphi\restriction_{\ell^\infty(\tau)}:\ell^\infty(\tau)\to\ell^\infty(\tau)$ is not compact.

Now suppose $\varphi(\tau)$ is finite, in this case $\sigma_\varphi(\ell^\infty(\tau))$ is a finite dimensional subset of $\sigma_\varphi(\ell^\infty(\tau))$, thus its closed bounded subsets are compact, using Theorem 1, $\sigma_\varphi\{x\in\ell^\infty(\tau):\|x\|_\infty<1\}(\subseteq\{x\in\ell^\infty(\tau):\|x\|_\infty<1\})$ is a bounded subset of $\sigma_\varphi(\ell^\infty(\tau))$, which leads to the desired result.

3. Generalized shifts on subspaces of ℓ^∞

As it is common in the literature, for the least infinite cardinal number $\omega=\{0,1,2,\ldots\}$ we denote $\ell^\infty(\omega)$ by ℓ^∞. Consider the following subspaces of ℓ^∞:

- $\ell^\infty_{00}:=\{(x_n)_{n<\omega}\in\ell^\infty:\exists N\ \forall n\ge N\ x_n=0\}$
- $\ell^\infty_{0c}:=\{(x_n)_{n<\omega}\in\ell^\infty:\exists z\ \exists N\ \forall n\ge N\ x_n=z\}$
- $\ell^\infty_{0}:=\{(x_n)_{n<\omega}\in\ell^\infty:\lim_{n\to+\infty}x_n=0\}$
- $\ell^\infty_{c}:=\{(x_n)_{n<\omega}\in\ell^\infty:\exists z\ \lim_{n\to+\infty}x_n=0\}$

thus $\ell^\infty_{00}\subseteq\ell^\infty_0\subseteq\ell^\infty_c\subseteq\ell^\infty$ and $\ell^\infty_{00}\subseteq\ell^\infty_{0c}\subseteq\ell^\infty_c\subseteq\ell^\infty$. In this section consider $\varphi:\omega\to\omega$.

Theorem 3. The following statements are equivalent:

1. $\sigma_\varphi(\ell^\infty_{00})\subseteq\ell^\infty_{00}$,
2. $\sigma_\varphi(\ell^\infty_{0})\subseteq\ell^\infty_{0}$,
3. for all $n\in\omega$ the set $\varphi^{-1}(n)$ is finite (i.e., φ is finite fiber).

Proof. "(2)$\Rightarrow$(3)" and "(1)$\Rightarrow$(3)": Suppose there exists $p\in\omega$ such that $\varphi^{-1}(p)$ is infinite. Consider $u=(u_n)_{n<\omega}$ with $u_p=1$ and $u_n=0$ for $n\ne p$. Then we have $u\in\ell^\infty_{00}(=\ell^\infty_0\cap\ell^\infty_{00})$ and $\sigma_\varphi(u)\notin\ell^\infty_0(=\ell^\infty_0\cup\ell^\infty_{00})$, thus not only $\sigma_\varphi(\ell^\infty_0)\not\subset\ell^\infty_0$, but also $\sigma_\varphi(\ell^\infty_{00})\not\subset\ell^\infty_{00}$.

(3)$\Rightarrow$(1): Suppose (3) is valid and $(x_n)_{n<\omega}\in\ell^\infty_{00}$, then there exists $N\in\omega$ such that for all $n\ge N$ we have $x_n=0$. Since φ is finite fiber, $\varphi^{-1}(\{0,\ldots,N\})$ is finite and $m=\max(\varphi^{-1}(\{0,\ldots,N\})\cup\{0\})\in\omega$. So $x_{\varphi(n)}=0$ for all $n\ge m+1$. Hence $\sigma_\varphi((x_n)_{n<\omega})=(x_{\varphi(n)})_{n<\omega}\in\ell^\infty_{00}$.

(3)$\Rightarrow$(2): Suppose (3) is valid and $(x_n)_{n<\omega}\in\ell^\infty_0$, then $\lim_{n\to+\infty}x_n=0$ and for every $\varepsilon>0$ there exists $N\in\omega$ such that for all $n\ge N$ we have $|x_n|<\varepsilon$. Since φ is finite fiber, $m=\max(\varphi^{-1}(\{0,\ldots,N\})\cup\{0\})\in\omega$. So for all $n\ge m+1$ we have $|x_{\varphi(n)}|<\varepsilon$. Thus $\lim_{n\to+\infty}x_{\varphi(n)}=0$ and $\sigma_\varphi((x_n)_{n<\omega})=(x_{\varphi(n)})_{n<\omega}\in\ell^\infty_0$.

Theorem 4. The following statements are equivalent:

1. $\sigma_\varphi(\ell^\infty_{0c})\subseteq\ell^\infty_{0c}$,
2. $\sigma_\varphi(\ell^\infty_{c})\subseteq\ell^\infty_{c}$,
3. for all $n\in\omega$ "$\varphi^{-1}(n)$ is finite" or "$\omega\setminus\varphi^{-1}(n)$ is finite".

Proof. First suppose there exists $p\in\omega$ such that both sets $\varphi^{-1}(p)$ and $\omega\setminus\varphi^{-1}(p)$ are infinite. Consider $u=(u_n)_{n<\omega}$ with $u_p=1$ and $u_n=0$ for $n\ne p$. Then we have $u\in\ell^\infty_{0c}(=\ell^\infty_c\cap\ell^\infty_{0c})$, let $(v_n)_{n<\omega}=(u_{\varphi(n)})_{n<\omega}=\sigma_\varphi(u)$. Using infiniteness of $\varphi^{-1}(p)$ and $\omega\setminus\varphi^{-1}(p)$ there exist $m_1<m_2<\cdots$ in $\varphi^{-1}(p)$ and there exist $k_1<k_2<\cdots$ in $\omega\setminus\varphi^{-1}(p)$ thus $\lim_{n\to\infty}v_{m_n}=1$ and $\lim_{n\to\infty}v_{k_n}=0$. Hence $\lim_{n\to\infty}v_n$ does not exist and $\sigma_\varphi(u)=(v_n)_{n<\omega}\notin\ell^\infty_c$. So not only $\sigma_\varphi(\ell^\infty_0)\not\subset\ell^\infty_0$, but also $\sigma_\varphi(\ell^\infty_{00})\not\subset\ell^\infty_{00}$. Thus "(2)$\Rightarrow$(3)" and "(1)$\Rightarrow$(3)".

(3) $\Rightarrow$ (1): Suppose (3) is valid and $(x_n)_{n<\omega} \in \ell_{0c}^{\infty}$, then there exists $N \in \omega$ such that for all $n \geq N$ we have $x_n = x_N =; z$. We have the following cases:

Case 1: φ is finite fiber. In this case $\varphi^{-1}(\{0,\ldots,N\})$ is finite and $m = \max(\varphi^{-1}(\{0,\ldots,N\}) \cup \{0\}) \in \omega$. So $x_{\varphi(n)} = z$ for all $n \geq m+1$. Hence $\sigma_{\varphi}((x_n)_{n<\omega}) = (x_{\varphi(n)})_{n<\omega} \in \ell_{0c}^{\infty}$.

Case 2: there exists $p \in \omega$ such that $\varphi^{-1}(p)$ is infinite. So in this case $\omega \setminus \varphi^{-1}(p)$ is finite and there exists $M \in \omega$ with $\omega \setminus \varphi^{-1}(p) \subseteq \{0,\ldots,M\}$. For all $n \geq M+1$ we have $n \in \varphi^{-1}(p)$ and $\varphi(n) = p$, hence $x_{\varphi(n)} = x_p$ which shows $\sigma_{\varphi}((x_n)_{n<\omega}) = (x_{\varphi(n)})_{n<\omega} \in \ell_{0c}^{\infty}$.

(3) $\Rightarrow$ (2): Suppose (3) is valid and $(x_n)_{n<\omega} \in \ell_c^{\infty}$, then $\{x_n\}_{n<\omega}$ is a convergent and hence Cauchy so for every $\varepsilon > 0$ there exists $N \in \omega$ such that for all $n,m \geq N$ we have $|x_n - x_m| < \varepsilon$. We have the following cases:

Case 1: φ is finite fiber. In this case $\varphi^{-1}(\{0,\ldots,N\})$ is finite and $M = \max(\varphi^{-1}(\{0,\ldots,N\}) \cup \{0\}) \in \omega$. So for all $n,m \geq M+1$ we have $\varphi(n),\varphi(m) > N$ therefore $|x_{\varphi(n)} - x_{\varphi(m)}| < \varepsilon$.

Case 2: there exists $p \in \omega$ such that $\varphi^{-1}(p)$ is infinite. So in this case $\omega \setminus \varphi^{-1}(p)$ is finite and there exists $M \in \omega$ with $\omega \setminus \varphi^{-1}(p) \subseteq \{0,\ldots,M\}$. For all $n,m \geq M+1$ we have $x_{\varphi(n)} = x_p = x_{\varphi(m)}$ which shows $|x_{\varphi(n)} - x_{\varphi(m)}| = 0 < \varepsilon$.

Using the above cases, there exists $M \in \omega$ with $|x_{\varphi(n)} - x_{\varphi(m)}| < \varepsilon$ for all $n,m \geq M+1$. Therefore $\{x_{\varphi(n)}\}_{n<\omega}$ is a Cauchy hence convergent sequence in $\mathbf{C}$. Therefore $\sigma_{\varphi}((x_n)_{n<\omega}) = (x_{\varphi(n)})_{n<\omega} \in \ell_c^{\infty}$.

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