Property (ao) AND TENSOR PRODUCT

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ABSTRACT: Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$ are a continuous linear operators and both have property (ao) then their tensor product has property (ao) if and only if the upper Weyl spectrum identity $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ holds true. Perturbations by quasi-nilpotent operators are considered.

1. INTRODUCTION

We will postulate along this paper X is a Banach space and BL(X) refer to each a continuous linear operators on X. For $\mathcal{S} \in \mathrm{BL(X)}$, let $\sigma(\mathcal{S}), \sigma_a(\mathcal{S})$ and iso $\sigma(\mathcal{S})$denote respectively the spectrum, the approximate point spectrum and isolated points of $\sigma(\mathcal{S})$. Let $\alpha(\mathcal{S})$ refer to the nullity of $\mathcal{S}$ defined by $\alpha(\mathcal{S}) = \dim \ker(\mathrm{S})$ and $\beta(\mathcal{S})$ refer to the deficiency of $\mathcal{S}$ defined by $\beta(\mathcal{S}) = \mathrm{codim}\, \mathrm{S(X)}$. If nullity of $\mathcal{S}$ is finite and rang of $\mathcal{S}$ ($\Re(\mathrm{S})$) is closed then $\mathcal{S}$ is called an upper semi-Fredholm operator and if deficiency of $\mathcal{S}$ is finite then $\mathcal{S}$ is a lower semi-Fredholm operator.

In the complete $\varphi_+(\mathrm{X})$ (resp. $\varphi_-(\mathrm{X})$) denote the set of all upper (resp. lower) semi-Freadholm operators on X. A continuous linear operator $\mathcal{S}$ is either upper or lower semi-Fredholm then $\mathcal{S}$ is semi-Fredholm (symbolizes $\varphi_+^-(\mathrm{X})$). While $\mathcal{S}$ is called a Freadholm operator (symbolizes $\varphi(\mathrm{X})$) if nullity and deficiency of $\mathcal{S}$ are finite. Now we can introduce the definition of an upper Weyl spectrum of $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \notin \varphi_+^-(\mathrm{X})\}$. $\mathrm{ind}(\mathcal{S})$ pointing to the index of $\mathcal{S}$ and defined as follows $\mathrm{ind}(\mathcal{S}) = \alpha(\mathcal{S}) - \beta(\mathcal{S})$. The ascent of $\mathcal{S} \in \mathrm{BL(X)}$ is littlest non-negative integer $\mathcal{p} = \mathcal{p}(\mathcal{S})$ such that $\ker \mathcal{S}^{\mathcal{p}} = \ker \mathcal{S}^{\mathcal{p}+1}$, if there is not such integer then $\ker \mathcal{S}^{\mathcal{p}} \neq \ker \mathcal{S}^{\mathcal{p}+1}$ for each $\mathcal{p}$, then $\mathcal{p}(\mathcal{S})$ is infinite. And the descent of an operator $\mathcal{S}$ is littlest non-negative integer $\mathcal{q} = \mathcal{q}(\mathcal{S})$ such that $\mathcal{S}^{\mathcal{q}}(\mathrm{X}) = \mathcal{S}^{\mathcal{q}+1}(\mathrm{X})$, if there is not such integer $\mathcal{S}^{\mathcal{q}}(\mathrm{X}) \neq \mathcal{S}^{\mathcal{q}+1}(\mathrm{X})$ for each $\mathcal{q}$ then $\mathcal{q}(\mathcal{S})$ is infinite. According to [1], the ascent and the descent are equal if $\mathcal{p}(\mathcal{S})$ and $\mathcal{q}(\mathcal{S})$ are finite.

A continuous linear operator $\mathcal{S} \in \mathrm{BL(X)}$ is Weyl if $\mathcal{S}$ is Fredholm of index zero, whilst is said to be Browder if $\mathcal{S} \in \varphi(\mathrm{X})$ and $\mathcal{p}(\mathcal{S}), \mathcal{q}(\mathcal{S})$ are finite. The Weyl, Browder and Browder approximate point spectrum define as follows

$$\sigma_W(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not Weyl}\},$$

$\sigma_b(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not Browder}\}$,

$\sigma_{ab}(\mathcal{S}) = \{\eta \in \sigma_a(\mathcal{S}): \eta \notin \varphi_+(\mathcal{A}) \text{ and } \mathcal{p}(\mathcal{S} - \eta) = \infty\}$.

An operator $\mathcal{S} \in \mathrm{BL(X)}$ is satisfies Weyl's Theorem if $\sigma(\mathcal{S}) \backslash \sigma_W(\mathcal{S}) = \mathrm{E}^0(\mathcal{S})$ and satisfies Browder's Theorem if $\sigma(\mathcal{S}) \backslash \sigma_W(\mathcal{S}) = \Pi^0(\mathcal{S})$ where $\mathrm{E}^0(\mathcal{S})$ is the eigenvalue of finite multiplicity and $\Pi^0(\mathcal{S})$ is poles of $\mathcal{S}$. We can say also a-Weyl's Theorem holds for $\mathcal{S}$ if $\sigma_a(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \mathrm{E}_a^0(\mathcal{S})$ and a-Browder's Theorem holds for $\mathcal{S}$ if $\sigma_a(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \Pi_a^0(\mathcal{S})$ where $\mathrm{E}_a^0(\mathcal{S})$ an eigenvalue of $\mathcal{S}$ of finite multiplicity that isolated in approximate point spectrum of $\mathcal{S}$ and $\Pi_a^0(\mathcal{S})$ is left poles of $\mathcal{S}$ of finite rank.

And we continuous to narrate the theories, but before this we will impose n is non-negative integer and $\mathcal{S} \in \mathrm{BL(X)}$ define $\mathcal{S}_{[n]}$ to be restriction of $\mathcal{S}$ to $\Re(\mathcal{S}^n)$ are seen as a map from $\Re(\mathcal{S}^n)$ into $\Re(\mathcal{S}^n)$, [special case $\mathcal{S}_{[0]} = \mathcal{S}$]. For some integer n, if the rang space $\Re(\mathcal{S}^n)$ is closed and $\mathcal{S}_{[n]}$ is an upper semi- Fredholm operator, then $\mathcal{S}$ is said to be upper semi B-Fredholm, while if the rang space $\Re(\mathcal{S}^n)$ is closed and $\mathcal{S}_{[n]}$ is a lower semi-Fredholm operator, then $\mathcal{S}$ is called lower semi B $-$Fredholm. The index of $\mathcal{S}$ is defined as the index of operator.

For $\mathcal{S} \in \mathrm{BL(X)}$, is called B $-$Weyl if it a B $-$Fredholm operator of index zero, and so B $-$Weyl spectrum of $\mathcal{S}$ is defined by $\sigma_{Bw}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not B} - \text{Weyl}\}$. So we can say that an operator $\mathcal{S}$ achieves generalized Weyl's Theorem if $\sigma(\mathcal{S}) \backslash \sigma_{Bw}(\mathcal{S}) = \mathrm{E}(\mathcal{S})$, and achieves generalized Browder's Theorem if $\sigma(\mathcal{S}) \backslash \sigma_{Bw}(\mathrm{S}) = \Pi(\mathcal{S})$, where E(S) is an eigenvalue of $\mathcal{S}$ that are isolated in spectrum of $\mathcal{S}$ and Π(S) is a poles of resolvent of S. The class of all upper semi B $-$Fredholm operators we will signal to him $\mathcal{SBF}_+(\mathrm{X})$ whereas $\mathcal{SBF}_+^-(\mathrm{X}) = \{\eta \in \mathcal{SBF}_+(\mathrm{X}): \mathrm{ind}(\mathcal{S}) \leq 0\}$, thus it will be defined the upper B $-$Weyl spectrum is $\sigma_{\mathcal{SBF}_+^-}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \notin \mathcal{SBF}_+^-(\mathrm{X})\}$. Hence after definition upper B $-$Weyl spectrum we call recall generalized a-Weyl's Theorem and generalized a-Browder's Theorem alternately, $, \sigma_a(\mathcal{S}) \backslash \sigma_{\mathcal{SBF}_+^-}(\mathcal{S}) = \mathrm{E}_a(\mathcal{S})$ and $\sigma_a(\mathcal{S}) \backslash \sigma_{\mathcal{SBF}_+^-}(\mathcal{S}) = \Pi_a(\mathcal{S})$, where $\mathrm{E}_a(\mathcal{S})$ is an eigenvalue of $\mathcal{S}$ that are isolated in approximate point spectrum of $\mathcal{S}$ and $\Pi_a(\mathcal{S})$ is a left poles of $\mathcal{S}$. Remain to mention the definition of Drazin spectrum and left Drazin invertible spectrum, if $\mathcal{S}$ has finite ascent and descent then $\mathcal{S}$ is called Drazin invertible, the Drazin spectrum $\sigma_D(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \text{ is not a Drazin invertible}\}$. An operator $\mathcal{S}$ is called left Drazin invertible (in symbol LD(X)), if $\mathrm{LD(X)} = \{\mathcal{S} \in \mathrm{BL(X)}: \mathcal{p}(\mathcal{S}) < \infty \text{ and } \Re(\mathcal{S}^{\mathcal{p}(\mathcal{S})+1}) \text{ is closed}\}$, and left Drazin invertible spectrum is defined by $\sigma_{LD}(\mathcal{S}) = \{\eta \in \mathbb{C}: \mathcal{S} - \eta \notin \mathrm{LD(X)}\}$.

Recall that a continuous linear operator $\mathcal{S} \in \mathrm{BL(X)}$, has single valued extension property at a point $\eta_0 \in \mathbb{C}$ (Shortly $\mathcal{SVEP}$), if for every open disc $\mathcal{U}$ centered at η_0 then only analytic function f: $\mathcal{U} \longrightarrow \mathcal{A}$ satisfying $(\mathcal{S} - \eta)\mathrm{f}(\eta) = 0$ is the function $\mathrm{f} \equiv 0$. Evidently, $\mathcal{S}$ has $\mathcal{SVEP}$ at every isolated point of the spectrum, consequently, note that the single valued extension property plays an important role in Fredholm and spectral Theory.

We postulate that $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, the tenors product of two operators $\mathcal{S}_1$and $\mathcal{S}_2$ on $\mathrm{X}_1 \otimes \mathrm{X}_2$ is the operator $\mathcal{S}_1 \otimes \mathcal{S}_2$ defined by $(\mathcal{S}_1 \otimes \mathcal{S}_2) \sum_i x_{1i} \otimes x_{2i} = \sum_i \mathcal{S}_1 x_{1i} \otimes \mathcal{S}_2 x_{2i}$ for all $\sum_i x_{1i} \otimes x_{2i} \in \mathrm{X}_1 \otimes \mathrm{X}_2$. [6,8], if $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy Browder's Theorem then $\mathcal{S}_1 \otimes \mathcal{S}_2$ satisfies Browder's Theorem if and only if the Weyl spectrum identity $\sigma_w(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_w(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_w(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ holds, and if $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy a-Browder's Theorem then $\mathcal{S}_1 \otimes \mathcal{S}_2$ satisfies a-Browder's Theorem if and only if the upper Weyl spectrum identity $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma_a(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma_a(\mathcal{S}_1)$ holds.

2. Property (ao) and tensor product

The most important findings of this paper is, if $\mathcal{S}_1 \in BL(X_1)$ and $\mathcal{S}_2 \in BL(X_2)$ have property (ao) then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao) if and only if the upper Weyl spectrum identity $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ holds, also study perturbation under a quasi-nilpotent operator for these royalty, this is part of the study, While the other is assume that $\mathcal{S}_1 \in BL(X_1)$ and $\mathcal{S}_2 \in BL(X_2)$ are polaroid and $\mathcal{S}_1^*$, $\mathcal{S}_2^*$ have $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (SZ), and study perturbation by commutator a quasi-nilpotent operator for property (SZ). The following lemmas help to reach the desired results: [1, Theorem 3.23], If $\mathcal{S} \in BL(X)$ has $\mathcal{SVEP}$ at $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+}(\mathcal{S})$ then $\eta \in \text{iso}\,\sigma_a(\mathcal{S})$ and $p(\mathcal{S} - \eta) < \infty$. From [4] and [11] we get the following results

i- $\sigma_x(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_x(\mathcal{S}_1)\sigma_x(\mathcal{S}_2)$, where $\sigma_x = \sigma$ or $\sigma_x = \sigma_a$,

ii $-$ $\sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+}(\mathcal{S}_1)\sigma_a(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+}(\mathcal{S}_2)\sigma_a(\mathcal{S}_1)$,

iii $-$ $\sigma_{\mathcal{SF}_-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_-}(\mathcal{S}_1)\sigma_\delta(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+}(\mathcal{S}_2)\sigma_\delta(\mathcal{S}_1)$.

and proposition 3 in [12], we obtain $\text{iso}\,\sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \subset \text{iso}\,\sigma(\mathcal{S}_1)\,\text{iso}\,\sigma(\mathcal{S}_2)$.

Lemma 2.1 Let $\mathcal{S}_1$, $\mathcal{S}_2$ are a continuous linear operators in $BL(X_1)$ and $BL(X_2)$ respectively, then $0 \notin \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \backslash\ \sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

proof: We assume that $0 \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$ that is $\mathcal{S}_1 \otimes \mathcal{S}_2$ is not invertible and therefore $0 \in \text{iso}\,\sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and from [1, Theorem 3.18], $\mathcal{S}_1 \otimes \mathcal{S}_2$ has $\mathcal{SVEP}$. And $0 \notin \sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2)$, so that $\mathcal{S}_1 \otimes \mathcal{S}_2$ has closed rang and $0 < \alpha(\mathcal{S}_1 \otimes \mathcal{S}_2) < \infty$. Since $\mathcal{S}_1 \otimes \mathcal{S}_2$ is surjective and has $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ is injective [1, corollary 2.24], consequently $\mathcal{S}_1$and $\mathcal{S}_2$ are injective if and only if $\mathcal{S}_1 \otimes \mathcal{S}_2$ is injective, we obtain $\alpha(\mathcal{S}_1) > 0$ or $\alpha(\mathcal{S}_2) > 0$. But $\alpha(\mathcal{S}_1 \otimes \mathcal{S}_2)$ is infinite, this leads to a discrepancy

Lemma 2.2 Let $\mathcal{S}_1 \in BL(X_1)$ and $\mathcal{S}_2 \in BL(X_2)$, then $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$

$$\subseteq \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1) = \sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2).$$

Proof: The inclusion $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1) \subseteq \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ verified because $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) \subseteq \sigma_{ab}(\mathcal{S})$ for all operator $\mathcal{S}$. Now we must prove that $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$, let $\eta \notin \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ as $\sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ implies that $\eta \neq 0$. Presume $\eta = \hbar\ell$ be any factorization of η, we obtain $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$ and therefor $\hbar \in \sigma(\mathcal{S}_1) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)$. Then $\hbar \in \varphi_+(\mathcal{S}_1)$, $\text{ind}\,(\mathcal{S}_1 - \hbar) \leq 0$, and $\ell \in \varphi_+(\mathcal{S}_2)$, $\text{ind}\,(\mathcal{S}_2 - \ell) \leq 0$. Consequently, $\eta \notin \sigma_{\mathcal{SF}_+}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. The following requirement is proven $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) \leq 0$, assume $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) > 0$, then $\alpha(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$ and so $\beta(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$ thus $\eta \in \varphi(\mathcal{S}_1 \otimes \mathcal{S}_2)$. Let $\Lambda = \{(\hbar_i, \ell_i)_{i=1}^p \in \sigma(\mathcal{S}_1)\sigma(\mathcal{S}_2) : \hbar_i \ell_i = \eta\}$, where Λ is a finite set. And calculate $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta)$ we will use Theorem 3.5 in [10], whereas $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) = \sum_{j=n+1}^p \text{ind}(\mathcal{S}_1 - \hbar_j) \dim H_0\,(\mathcal{S}_2 - \ell_j) + \sum_{j=1}^n \text{ind}(\mathcal{S}_2 - \ell_j) \dim H_0\,(\mathcal{S}_1 - \hbar_j)$, since $\text{ind}(\mathcal{S}_1 - \hbar_i)$ and $\text{ind}(\mathcal{S}_2 - \ell_i)$ are non-positive, This is competitive. And so $\text{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) \leq 0$ thus $\eta \notin \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

Rest to prove $\sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$. Let $\eta \notin \sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2)$ then $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $p(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$ implies that $\eta \in \text{iso}\,\sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$. For all factorization $\eta = \hbar\ell$ of η such that $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$ that is $\hbar \in \varphi_+(\mathcal{S}_1)$ and $\ell \in \phi_+(\mathcal{S}_2)$. As $\text{iso}\,\sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \subset \text{iso}\,\sigma(\mathcal{S}_1)\,\text{iso}\,\sigma(\mathcal{S}_2)$, then $\mathcal{S}_1$ has $\mathcal{SVEP}$ at $\hbar$ and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at ℓ. Thus we have $p(\mathcal{S}_1 - \hbar) < \infty$ and $p(\mathcal{S}_2 - \ell) < \infty$, therefore $\hbar \notin \sigma_{ab}(\mathcal{S}_1)$ and $\ell \notin \sigma_{ab}(\mathcal{S}_2)$ and so $\eta \notin \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$.

We postulated $\eta \notin \sigma_{ab}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{ab}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$, since $\eta \neq 0$for any factorization $\eta = \hbar\ell$ of η such that $\hbar \in \sigma(\mathcal{S}_1)$, $\ell \in \sigma(\mathcal{S}_2)$ and $\hbar \notin \sigma_{ab}(\mathcal{S}_1)$, $\ell \notin \sigma_{ab}(\mathcal{S}_2)$, then $\hbar \in \varphi_+(\mathcal{S}_1)$, $p(\mathcal{S}_1 - \hbar) < \infty$ and $\ell \in \varphi_+(\mathcal{S}_2)$, $p(\mathcal{S}_2 - \ell) < \infty$, implies that $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $\hbar \in \text{iso}\,\sigma(\mathcal{S}_1)$, $\ell \in \text{iso}\sigma\,(\mathcal{S}_2)$, that is $\eta \in \text{iso}\,\sigma(\mathcal{S}_1 \otimes \mathcal{S}_2)$. It follows that $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $p(\mathcal{S}_1 \otimes \mathcal{S}_2 - \eta) < \infty$. Hence $\eta \notin \sigma_{ab}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. So we get the result.

Definition 2.3 [3] A continuous linear operator $\mathcal{S} \in \mathcal{L}(\mathcal{A})$ is said to have property (ao) if $\sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \Pi_a(\mathcal{S})$.

Proposition 2.4 Let $\mathcal{S}$ be a continuous linear operators that the following are equivalent for $\mathcal{S}$

i- property (ao)holds for $\mathcal{S}$,

ii- $\sigma_{ab}(\mathcal{S}) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$.

Proof: For every operators $\mathcal{S}$, $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) \subseteq \sigma_{ab}(\mathcal{S})$. Let $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$, since property (ao) holds for $\mathcal{S}$ then $\eta \in \Pi_a(\mathcal{S})$. But by Theorem [3], property (Sab) holds for $\mathcal{S}$ then $\eta \in \Pi_a^0(\mathcal{S})$ while that $\Pi_a^0(\mathcal{S}) = \sigma_a(\mathcal{S}) \backslash \sigma_{ab}(\mathcal{S})$. Therefore $\sigma_{ab}(\mathcal{S}) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$.

Reciprocally, let $\eta \in \Pi_a(\mathcal{S})$, that is $\eta \in \sigma_a(\mathcal{S})$ and $\eta \notin \sigma_{LD}(\mathcal{S})$. But $\sigma_a(\mathcal{S}) \subseteq \sigma(\mathcal{S})$ and $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}) \subseteq \sigma_{LD}(\mathcal{S})$, then we get $\eta \in \sigma(\mathcal{S})$ and $\eta \notin \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$. Thus $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$. Now, let $\eta \in \sigma(\mathcal{S}) \backslash \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$. Since $\sigma_{ab}(\mathcal{S}) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S})$ then property (az) holds for $\mathcal{S}$ and therefore $\eta \in \Pi_a^0(\mathcal{S})$. As $\Pi_a^0(\mathcal{S}) \subseteq \Pi(\mathcal{S})$, then $\eta \in \Pi_a(\mathcal{S})$. Consequently, property (ao) holds for $\mathcal{S}$.

The following Theorem proves that the above lemma validates for two directions if we add the condition $\mathcal{S}_1$ has property (ao) and $\mathcal{S}_2$ has property (ao).

Theorem 2.5 Suppose that $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$, and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, and both have property (ao), then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao) if and only if $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$.

Proof: We assume that $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao)then by above lemma we get the result.

Reciprocally, Since $\mathcal{S}_1$, $\mathcal{S}_2$ has property (ao) then $\sigma_{\mathrm{ab}}(\mathcal{S}_1) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)$, $\sigma_{\mathrm{ab}}(\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)$. According to the hypothesis

$$\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1) = \sigma_{\mathrm{ab}}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathrm{ab}}(\mathcal{S}_2)\sigma(\mathcal{S}_1) =$$

$\sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2)$, thus $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao).

Theorem 2.6 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ have property (ao). Then $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$ if and only if $\mathcal{S}_1$ has $\mathcal{SVEP}$ at every points $\hbar \in \varphi_+(\mathcal{S}_1)$and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at every points $\ell \in \phi_+(\mathcal{S}_2)$ such that $0 \neq \eta = \hbar\ell \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

Proof: We assume that $\eta \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$ then $\eta \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2)$, because $\mathcal{S}_1$ and $\mathcal{S}_2$ have property (ao). For every factorization $0 \neq \eta = \hbar\ell$ of η such that $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$, we have $\hbar \in \varphi_+(\mathcal{S}_1)$ and $\ell \in \varphi_+(\mathcal{S}_2)$ And consequently $p(\mathcal{S}_1 - \hbar) < \infty$ and $p(\mathcal{S}_2 - \ell) < \infty$. It leads to $\mathcal{S}_1$ has $\mathcal{SVEP}$ at $\hbar$ and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at ℓ.

Reciprocally, we must prove that $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1)$. Enough to prove that $\sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2) \subseteq \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. Let $\eta \in \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$ then $\eta \in \varphi_+(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and $\mathrm{ind}(\mathcal{S}_1 \otimes \mathcal{S}_2) \leq 0$. Hence for every factorization $0 \neq \eta = \hbar\ell$ of η where $\hbar \in \sigma(\mathcal{S}_1)$ and $\ell \in \sigma(\mathcal{S}_2)$, and $\hbar \in \varphi_+(\mathcal{S}_1)$, $\ell \in \varphi_+(\mathcal{S}_2)$. Since $\mathcal{S}_1$ has $\mathcal{SVEP}$ at $\hbar$ and $\mathcal{S}_2$ has $\mathcal{SVEP}$ at ℓ then $p(\mathcal{S}_1 - \hbar) < \infty$ and $p(\mathcal{S}_2 - \ell) < \infty$. Therefore $\hbar \notin \sigma_{\mathrm{ab}}(\mathcal{S}_1)$ and $\ell \notin \sigma_{\mathrm{ab}}(\mathcal{S}_2)$. Thus $\eta \notin \sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2)$.

Theorem 2.7 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be continuous linear operator in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively. If $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (ao).

Proof: As $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$ then satisfy generalized a-Browder Theorem and consequently $\mathcal{S}_1$, $\mathcal{S}_2$ satisfy a-Browder Theorem. Then by Theorem 1 in [8], a-Browder Theorem holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$, Thus $\sigma_{\mathrm{ab}}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. It leads to property (ao)holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$.

Theorem 2.8 Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, If $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$ then $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ has property (ao).

Proof: As $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$ then we obtain by [1, corollary 3.73], $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ obey a-Browder Theorem. Consequently, $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ obey a-Browder Theorem. That is $\sigma_{\mathrm{ab}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*)$. Evidently, $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ obey property (ao).

Duggal in [5, 9]defined the polaroid operator as follows, if every isolated point of the spectrum of $\mathcal{S}$ is the pole of resolvent of $\mathcal{S}$, also η is pole of resolvent of $\mathcal{S}$ if and only if $0 < p(\mathcal{S} - \eta) = q(\mathcal{S} - \eta) < \infty$. Or equivalent, an operator $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$ is called polaroid if and only if there exists $\mathrm{d} = \mathrm{d}(\eta) \in \mathbb{N}$ such that $\mathrm{H}_0(\mathcal{S} - \eta) = \ker(\mathcal{S} - \eta)^{-1}$, for all $\eta \in \mathrm{iso}\sigma(\mathcal{S})$. Where $\mathrm{H}_0(\mathcal{S} - \eta)$ is a quasi-nilpotent part of $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$define as follows $\mathrm{H}_0(\mathcal{S} - \eta) = \{\mathrm{a} \in \mathrm{X}: \lim_{n\to\infty} \| (\mathcal{S} - \eta)^n \mathrm{a} \|^{\frac{1}{n}} = 0\}$.

Definition 2.9 [3] A continuous linear operator $\mathcal{S} \in \mathrm{BL}(\mathrm{X})$is said to have property (SZ) if $\sigma(\mathcal{S}) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}) = \mathrm{E}(\mathcal{S})$.

Theorem 2.10 Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$ are polaroid. If $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$ then $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (SZ).

Proof: Let's start with the imposition $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ have $\mathcal{SVEP}$, then we have

$$\sigma_{\mathrm{W}}(\mathcal{S}_1) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1) = \sigma_{\mathrm{Bw}}(\mathcal{S}_1)$$

$$\sigma_{\mathrm{W}}(\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2) = \sigma_{\mathrm{Bw}}(\mathcal{S}_2),$$

also we have $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_1 \otimes \mathcal{S}_2$ satisfies Browder's Theorem, thus

$$\begin{aligned}\sigma_{\mathrm{b}}(\mathcal{S}_1 \otimes \mathcal{S}_2) &= \sigma_{\mathrm{W}}(\mathcal{S}_1 \otimes \mathcal{S}_2)\\ &= \sigma_{\mathrm{W}}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathrm{W}}(\mathcal{S}_2)\sigma(\mathcal{S}_1)\\ &= \sigma_{\mathrm{Bw}}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathrm{Bw}}(\mathcal{S}_2)\sigma(\mathcal{S}_1)\\ &= \sigma_{\mathrm{Bw}}(\mathcal{S}_1 \otimes \mathcal{S}_2)\\ &= \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1)\sigma(\mathcal{S}_2) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2)\sigma(\mathcal{S}_1) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2).\end{aligned}$$

As $\mathcal{S}_1$ and $\mathcal{S}_2$ are polaroid implies that $\mathcal{S}_1 \otimes \mathcal{S}_2$ is polaroid [6, Lemma 2], and consequently Weyl's Theorem holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$. From [7, Theorem 3.17], generalized Weyl Theorem holds for $\mathcal{S}_1 \otimes \mathcal{S}_2$, thus $\sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathrm{Bw}}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \sigma(\mathcal{S}_1 \otimes \mathcal{S}_2) \setminus \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1 \otimes \mathcal{S}_2) = \mathrm{E}(\mathcal{S}_1 \otimes \mathcal{S}_2)$. Plainly, $\mathcal{S}_1 \otimes \mathcal{S}_2$ has property (SZ).

Theorem 2.11 Let $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$ are polaroid. If $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$ then $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ has property (SZ).

Proof: We assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ have $\mathcal{SVEP}$, then we have from [1, corollary 2.5], [1, corollary 3.53], [2, Theorem 2.20]

$$\sigma_{\mathrm{W}}(\mathcal{S}_1{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^*) = \sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^*)$$

$$\sigma_{\mathrm{W}}(\mathcal{S}_2{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2{}^*) = \sigma_{\mathrm{Bw}}(\mathcal{S}_2{}^*),$$

also we have $\mathcal{S}_1{}^*$, $\mathcal{S}_2{}^*$ and $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ satisfies a-Browder's Theorem and therefore Browder's Theorem , thus

$$\begin{aligned}\sigma_{\mathrm{b}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*) &= \sigma_{\mathrm{W}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*)\\ &= \sigma_{\mathrm{W}}(\mathcal{S}_1{}^*)\sigma(\mathcal{S}_2{}^*) \cup \sigma_{\mathrm{W}}(\mathcal{S}_2{}^*)\sigma(\mathcal{S}_1)\\ &= \sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^*)\sigma(\mathcal{S}_2{}^*) \cup \sigma_{\mathrm{Bw}}(\mathcal{S}_2{}^*)\sigma(\mathcal{S}_1{}^*)\\ &= \sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*)\\ &= \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^*)\sigma(\mathcal{S}_2{}^*) \cup \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_2{}^*)\sigma(\mathcal{S}_1{}^*) = \sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*).\end{aligned}$$

As $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ are polaroid implies that $\mathcal{S}_1{}^* \otimes \mathcal{S}_2{}^*$ is polaroid [6, Lemma 2], and consequently Weyl's

Theorem holds for $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$. From [7, Theorem 3.17], generalized Weyl Theorem holds for $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$, thus $\sigma(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*)\backslash\sigma_{\mathrm{Bw}}(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*) = \sigma(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*)\backslash\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*) = \mathrm{E}(\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*)$. Plainly, $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$ has property (SZ).

3. PERTURBATIONS

Assume $[\mathcal{Q},\mathcal{S}] = \mathcal{Q}\mathcal{S} - \mathcal{S}\mathcal{Q}$ refer to the commutator of operators $\mathcal{Q},\mathcal{S} \in \mathrm{BL(X)}$. We assume that $\mathcal{Q}_1, \mathcal{Q}_2$ in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively, are a quasi-nilpotent operators $[\mathcal{Q}_1,\mathcal{S}_1] = [\mathcal{Q}_2,\mathcal{S}_2] = 0$ for some operators $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_2)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$, hence $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2) = (\mathcal{S}_1\otimes\mathcal{S}_2) + \mathcal{Q}$, such that $\mathcal{Q} = \mathcal{S}_1\otimes\mathcal{Q}_1 + \mathcal{S}_2\otimes\mathcal{Q}_2 + \mathcal{Q}_1\otimes\mathcal{Q}_2 \in \mathrm{BL}(\mathrm{X}_1\otimes\mathrm{X}_2)$ is a quasi-nilpotent operator. Remember the definition of isoloid operator, $\mathcal{S} \in \mathrm{BL(X)}$, is isoloid if $\mathrm{iso}\,\sigma(\mathcal{S}) = \mathrm{E}(\mathcal{S})$.

Proposition 3.1 Suppose that $\mathcal{S} \in \mathrm{B(X)}$ be a polaroid operator then $\mathrm{E}(\mathcal{S}) = \Pi(\mathcal{S})$.

Proof: As always we have $\Pi(\mathcal{S}) \subseteq \mathrm{E}(\mathcal{S})$, for every operators $\mathcal{S}$. Now, let $\eta \in \mathrm{E}(\mathcal{S})$ that is $\eta \in \mathrm{iso}\,\sigma(\mathcal{S})$, since $\mathcal{S}$ is a polaroid then $\eta \in \Pi(\mathcal{S})$. Therefore $\mathrm{E}(\mathcal{S}) = \Pi(\mathcal{S})$.

Theorem 3.2 Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively, be a quasi-nilpotent operators $[\mathcal{Q}_1,\mathcal{S}_1] = [\mathcal{Q}_2,\mathcal{S}_2] = 0$ for some operators $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$. If $\mathcal{S}_1\otimes\mathcal{S}_2$ polaroid then property (ao) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$ implies $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)$ satisfies property (ao).

Proof: Observe that $\sigma(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, and that the perturbation of an operator by commuting quasi-nilpotent has $\mathcal{SVEP}$ if and only if the operator has $\mathcal{SVEP}$. If property (SZ) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$, hence

$\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)\backslash\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2)$

$\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)\backslash\sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big) = \Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2)$,

we ought prove that $\Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2) = \Pi_{\mathrm{a}}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Let $\eta \in \Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2)$, it leads to $\eta \in \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$ and $\eta \in \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, also $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$. Clearly, if $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$ hence $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$ has $\mathcal{SVEP}$ at η and therefore $\Pi_{\mathrm{a}}(\mathcal{S}_1\otimes\mathcal{S}_2) = \Pi(\mathcal{S}_1\otimes\mathcal{S}_2)$,also we have $(\mathcal{S}_1{}^* + \mathcal{Q}_1{}^*)\otimes(\mathcal{S}_2{}^* + \mathcal{Q}_2{}^*)$ has $\mathcal{SVEP}$ at η, Implies that $\eta \in \mathrm{iso}\,\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Since $\mathcal{S}_1\otimes\mathcal{S}_2$ be a polaroid it leads to $\mathcal{S}_1\otimes\mathcal{S}_2$ an isoloid then $\eta \in \mathrm{E}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, consequently by above proposition we get $\eta \in \Pi\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Therefore, $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)$ satisfies property (ao).

Theorem 3.3 Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ in $\mathrm{BL}(\mathrm{X}_1)$ and $\mathrm{BL}(\mathrm{X}_2)$ respectively, be a quasi-nilpotent operators $[\mathcal{Q}_1,\mathcal{S}_1] = [\mathcal{Q}_2,\mathcal{S}_2] = 0$ for some operators $\mathcal{S}_1 \in \mathrm{BL}(\mathrm{X}_1)$ and $\mathcal{S}_2 \in \mathrm{BL}(\mathrm{X}_2)$. If $\mathcal{S}_1\otimes\mathcal{S}_2$ isoloid then property (SZ) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$ implies $(\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)$ satisfies property (SZ).

Proof: Observe that $\sigma(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, $\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, and that the perturbation of an operator by commuting quasi-nilpotent has $\mathcal{SVEP}$ if and only if the operator has $\mathcal{SVEP}$. If property (SZ) holds for $\mathcal{S}_1\otimes\mathcal{S}_2$, hence

$\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)\backslash\sigma_{\mathcal{SF}_+^-}(\mathcal{S}_1\otimes\mathcal{S}_2) = \mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2)$

$\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)\backslash\sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big) = \mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2)$,

rest we prove that $\mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2) = \mathrm{E}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Let $\eta \in \mathrm{E}(\mathcal{S}_1\otimes\mathcal{S}_2)$, it leads to $\eta \in \sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$ and $\eta \notin \sigma_{\mathcal{SF}_+^-}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$, also $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$. Clearly, if $\eta \in \mathrm{iso}\,\sigma(\mathcal{S}_1\otimes\mathcal{S}_2)$ hence $\mathcal{S}_1{}^*\otimes\mathcal{S}_2{}^*$ has $\mathcal{SVEP}$ at η and therefore $(\mathcal{S}_1{}^* + \mathcal{Q}_1{}^*)\otimes(\mathcal{S}_2{}^* + \mathcal{Q}_2{}^*)$ has $\mathcal{SVEP}$ at η. Implies that $\eta \in \mathrm{iso}\,\sigma\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$. Since $\mathcal{S}_1\otimes\mathcal{S}_2$ isoloid then $\eta \in \mathrm{E}\big((\mathcal{S}_1 + \mathcal{Q}_1)\otimes(\mathcal{S}_2 + \mathcal{Q}_2)\big)$.

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