Study of a Predator-Prey Model with Modified Ratio-Dependent and Sokol-Howell Functional Response

Safaa Jawad Ali[1]
[1] Technical Instructor Preparing Institute, Middle Technical University, Baghdad, Iraq,
Corresponding addresses
safa_ali1962@yahoo.com

***Abstract*:** In this paper a predator-prey food chain model with modified ratio-dependent and Sokol-Howell functional response is proposed and discussed. The model is observed to be dissipative. The stability of the equilibrium points of the three species system is analyzed. The flow of the model is explored theoretically with two functional responses and numerically with three ones.

***Keywords*:** Sokol-Howell, modified ratio-dependent, stability analysis, functional response.

1. Introduction

As we all know that periodic and chaotic environmental models are eccentric in behavior. The permanence and extinction in predator-prey with ratio-dependent received attention by many ecological authors, see [1,3-6,11]. Jost and Arditi proves that prey and ratio-dependent systems can fit well with time arrangement created by each other [1]. Gakkhar and Naji in [6] studied the chaos in ratio-dependent model. Guin and Mandal [3] examined the flow of reaction-diffusion in ratio-dependent systems with intraspecific competition. Sokol-Howell functional response of the form $\frac{wx}{h+x^2}$ is studied by many ecologists; see [7-10]. In this paper, we modify the model of [8] by using the modified ratio-dependent Sokol-Howell functional response $\frac{wxy^2}{hy^2+x^2}$ in the place of the standard Sokol-Howell. The dynamics of the three-species predator-prey is studied (Stability analysis, Numerical exploration, results and conclusions), which shows the significance of the system beneath consideration.

2. The Mathematical Model

Consider the three species food chain model at time (t) consisting of the prey which is denoted by $x(t)$, the middle predator denoted by $y(t)$ and the top predator whose denoted by $z(t)$. The middle predator y preys on its only food x at the first level according to modified ratio-dependent Sokol-Howell functional response, while the top predator z preys on y at the second level according to the standard Sokol-Howell. The dynamics of the model can be represented by:

$$
\begin{aligned}
\frac{dx}{dt} &= a_1x - b_1x^2 - \frac{w_1xy^2}{h_1y^2+x^2} = G_1(x,y,z),\\
\frac{dy}{dt} &= \frac{w_2xy^2}{h_2y^2+x^2} - d_1y - \frac{w_3yz}{h_3+y^2} = G_2(x,y,z),\\
\frac{dz}{dt} &= \frac{w_4yz}{h_4+y^2} - d_2z = G_3(x,y,z). \qquad (1)
\end{aligned}
$$

The functional response in system (1) is proposed by removing the prey x and put the ratio $\frac{x}{y}$ in Sokol-Howell response. The solution of the system (1) exists and is unique since all the functions G_i $(i=1,2,3)$ are Lipschitzian on $R_+^3 = \{(x,y,z)\in R^3 : x\geq 0, y\geq 0, z\geq 0\}$. Here the positive constants a_1, b_1, d_j $(j=1,2)$ h_k and w_k $(k=1,2,3,4)$ denote to: a_1 is the growth rate of the prey x, b_1 represents the intraspecific competition of prey x, w_k's are the maximum values attainable by each per capita rate, h_k's are the half-saturation constant, d_j's represent the death rate of the middle and the top predators.

<u>Note</u>: System (1) is observed to be dissipative, see [8].

3. Stability Analysis

In this section, the stability of the equilibrium points of model (1) is discussed. The points $E_0=(0,0,0)$ and $E_1=(\frac{a_1}{b_1},0,0)$ are always exist. The third equilibrium point given by $E_2=(x_*,y_*,0)$ exists where

$$x_* = \frac{1}{b_1}\left(a_1 - \frac{w_2^3}{h_1w_2^2+4d_2^2h_2^2}\right) \quad \text{and} \quad y_* = \frac{w_2x}{2d_2h_2} \qquad (2)$$

with the following condition provided that $0 < x < \frac{a_1}{b_1}$

$$x^2(v_2 - 4d_2^2h_2) = 0. \qquad (3)$$

For the stability analysis of E_0, E_1 and E_2 see [8].

Now, the positive equilibrium point $E_3=(x^*,y^*,z^*)$ exists if there is appositive solution to the following equations in the $Int.R_+^3$.

$$g_1=a_1-b_1x-\frac{w_1y^2}{h_1y^2+x^2}=0,$$

$$g_2=\frac{w_2xy}{h_2y^2+x^2}-d_1-\frac{w_3z}{h_3+y^2}=0,$$

$$g_3=\frac{w_4y}{h_4+y^2}-d_2=0. \quad (4)$$

From the third equation of (4) we have

$$d_2y^2-w_4y+d_2h_4=0, \quad (5)$$

so that,

$$y^*=\frac{w_4\pm\sqrt{w_4^2-4d_2^2h_4}}{2d_2},$$

Hence, if the term $w_4^2-4d_2^2h_4<0$, then there is no positive solution to Eq. (5) and if $w_4^2-4d_2^2h_4>0$, then there are two positive solution to Eq. (5). The last case occurs if the following condition holds

$$w_4^2-4d_2^2h_4=0. \quad (6)$$

Then, there is only one solution given by

$$y^*=\frac{w_4}{2d_2}. \quad (7)$$

From the first equation of (4)

$$b_1x^{*3}-a_1x^{*2}+b_1h_1x^*y^{*2}+y^{*2}(w_1-a_1h_1)=0. \quad (8)$$

Equation (8) has one positive root depending on Descartes's rule if

$$w_1<a_1h_1. \quad (9)$$

Again, from the second equation of (4)

$$z^*=\frac{h_3+y^{*2}}{w_3}\left[\frac{w_2x^*y^*}{h_2y^{*2}+x^{*2}}-d_1\right], \quad (10)$$

Now, in addition to condition (6) and (9) the positive point E_3 exists if the following condition holds

$$\frac{w_2x^*y^*}{h_2y^{*2}+x^{*2}}>d_1. \quad (11)$$

The varational matrix $V=(x,y,z)$ is computed for system (4) as:

$$V(x,y,z)=\lfloor m_{ij}\rfloor \ i,j=1,2,3, \quad (12)$$

where

$$m_{11}=a_1-2b_1x^*-\frac{w_1y^{*2}(h_1y^{*2}-x^{*2})}{(h_1y^{*2}+x^{*2})^2}$$

$$m_{12}=-\frac{2w_1x^{*3}y^*}{(h_1y^{*2}+x^{*2})^2}$$

$$m_{13}=0,$$

$$m_{21}=\frac{w_2y^{*2}(h_2y^{*2}-x^{*2})}{(h_1y^{*2}+x^{*2})},$$

$$m_{22}=\frac{2w_2h_2x^*y^{*3}}{(h_1y^{*2}+x^{*2})^2}-d_1-\frac{w_3z^*(h_3-y^{*2})}{(h_3+y^{*2})^2},$$

$$m_{23}=-\frac{w_3y^*}{(h_3+y^{*2})},$$

$$m_{31}=0,$$

$$m_{32}=\frac{w_4z^*(h_4-y^{*2})}{(h_4+y^{*2})^2},$$

$$m_{33}=\frac{w_4y^*}{(h_4+y^{*2})}-d_2.$$

The characteristic equation of the above matrix (12) can be written as:

$$\lambda^3+H_1\lambda^2+H_2\lambda+H_3=0,$$

where

$$H_1=-(m_{11}+m_{22}+m_{33}),$$

$$=-\left(a_1-2b_1x^*-\frac{w_1y^{*2}(h_1y^{*2}-x^{*2})}{P_1^2}\right)$$

$$-\left(\frac{2w_2x^{*3}y^*}{P_2^2}-d_1-\frac{w_3z^*(h_3-y^{*2})}{Q_1^2}\right)$$

$$-\left(\frac{w_4y^*}{Q_2^2}-d_2\right),$$

where

$P_1=(h_1y^{*2}+x^{*2}),\ P_2=(h_2y^{*2}+x^{*2})$ and

$Q_1=(h_3+y^{*2}),\ Q_2=(h_4+y^{*2})$.

Similarly we write H_2, H_3 and $H_1H_2-H_3$ in the form of m_{ij}, where

$$H_2=(m_{11}m_{22}-m_{12}m_{21})+(m_{22}m_{33}-m_{23}m_{32})+m_{11}m_{33}$$

$$H_2=(m_{11}m_{22}-m_{12}m_{21})+m_{22}m_{33}+m_{11}m_{33}.$$

Since $y^{*2}=h_4$ according to condition (6).

$$H_3=m_{33}(m_{12}m_{21}-m_{11}m_{22}),$$

and

$$H_1H_2-H_3=-(m_{11}+m_{22})[(m_{11}m_{22}-m_{12}m_{21})+m_{22}m_{33}+m_{11}m_{33}]-m_{33}^2(m_{11}+m_{22})$$

Now, straightforward computations show that $H_1>0$, $H_3>0$, and $H_1H_2-H_3>0$ if and only if the next conditions are hold:

$$\frac{1}{2x^*}\left[a_1-\frac{w_1y^*(h_1y^{*2}-x^{*2})}{P_1^2}\right]<b_1, \quad (13)$$

$$h_1,\ h_2 > \frac{x^{*2}}{y^{*2}} \tag{14}$$

$$\frac{2w_2h_2x^*y^{*3}}{(h_1y^{*2}+x^{*2})^2} - \frac{w_3z^*(h_3-y^{*2})}{(h_3+y^{*2})^2} < d_1, \tag{15}$$

$$h_3 > y^{*2}, \tag{16}$$

$$\frac{w_4y^*}{Q_2} < d_2. \tag{17}$$

According to Routh-Hurwitz criterion, $E_3 = (x^*, y^*, z^*)$ is locally asymptotically stable in the $Int.R_+^3$ provided conditions (13-17) hold, see [2, 13].
Now, for the global asymptotic stability we didn't find a suitable Lyapunov function and we discuss the global dynamics numerically in the next section.

4. Numerical Exploration

The Runge-Kutta method of six order is used to solve the system (1) numerically, see [12]. There are two cases here to discuss. The first case of system (1) itself, and the second case we replacing the Sokol-Howell functional response by Leslie-Gower and we run the new system numerically so that to analyze the behavior of modified ratio-dependent functional response more.

4.1 Modified Ratio-Dependent with Sokol-Howell

For the following data set

$$a_1 = 0.20,\ \ b_1 = 0.0007,\ \ w_1 = 0.051,\ \ w_2 = 0.27,$$
$$w_3 = 0.21,\ \ w_4 = 0.095,\ \ d_1 = 0.0033,\ \ d_2 = 0.005,$$
$$h_1 = h_2 = 0.22,\ \ h_3 = 1.0. \tag{18}$$

The attractors for model (1) are plotted depending on the half-saturation constant h_4 of the top predator, since we discussed and other authors the effects of the growth rate, death rate and the intraspecific competition in many papers, see [3,8,9].

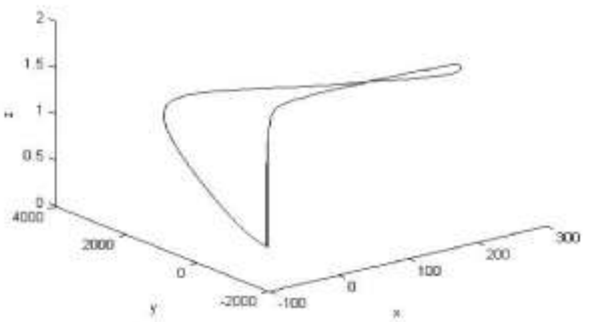

Figure 1. 3D of system (1) period 2 with data (18) and $h_4 = 1.20$ with fading in the top predator.

For h_4 with data (18), system (1) observed to be with period 2 and vanishing of the top predator as it shown in figure 1. Decreasing the value of h_4 from 0.9 to 0.5, then model (1) is periodic with period 1 as plotted in figure 2. Decreasing h_4 a little bit more for $h_4 = 0.4$, then system (1) food chain is stable as it shown in figure 3.

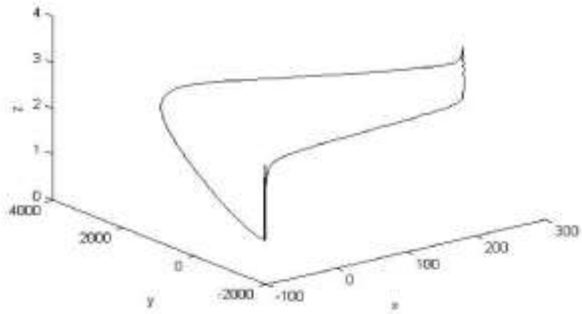

Figure 2. 3D of system (1) periodic with data (18) and $h_4 = 0.5$ with extinction in the top predator.

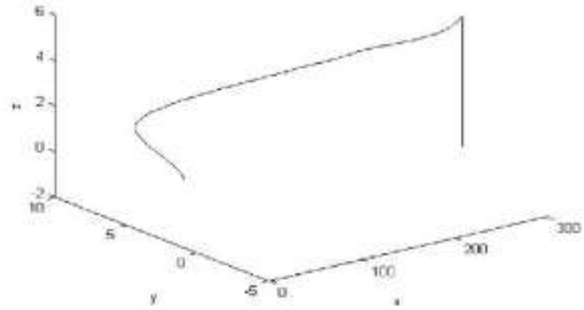

Figure 3. 3D of model (1) with data (18) stable for $h_4 = 0.4$

4.2 Modified Ratio-Dependent with Leslie-Gower

The food chain system (1) is modified numerically by putting the Leslie-Gower in the place of Sokol-Howell in third equation of system (1) and the top predator equation written as:

$$\frac{dz}{dt} = c_3z^2 - \frac{w_4z^2}{h_4+y}, \tag{19}$$

and also we don't forget to replace y^2 in the denominator of last term of the middle predator by y and the last term change to $\frac{w_3yz}{h_3+y}$. The model in [8], we used the standard Sokol-Howell with Leslie-Gower and the model exhibits chaotic dynamics. Now, for the following data set

$$a_1 = 2.50,\ \ b_1 = 0.5,\ \ w_1 = 0.25,\ \ w_2 = 7.5,$$
$$w_3 = 0.21,\ \ w_4 = 1.925,\ \ d_1 = 0.0042,\ \ c_3 = 0.005,$$
$$h_1 = h_2 = 20.0,\ \ h_3 = h_4 = 10.0, \tag{20}$$

We run the Lesile-Gower with modified ratio-dependent for data (20) and our target to see the changes in the behavior of the system dynamics and also comparing our results in section **5** with the model in [8].

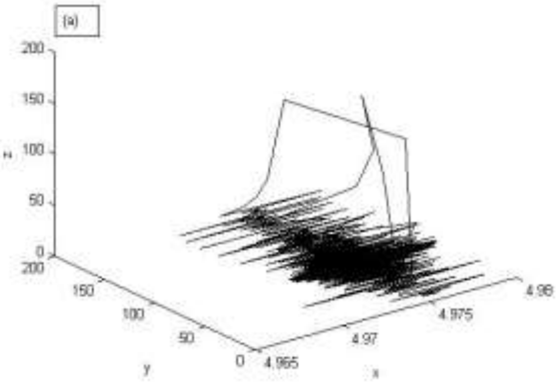

Figure 4(a). 3D of modified ratio-dependent and Leslie-Gower with data (20), stable with persistence of the prey x **,** middle predator y and the top predator z **.**

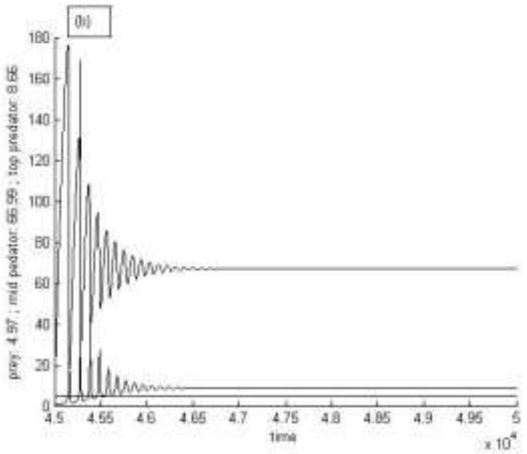

Figure 4(b). Time series of figure (5a).

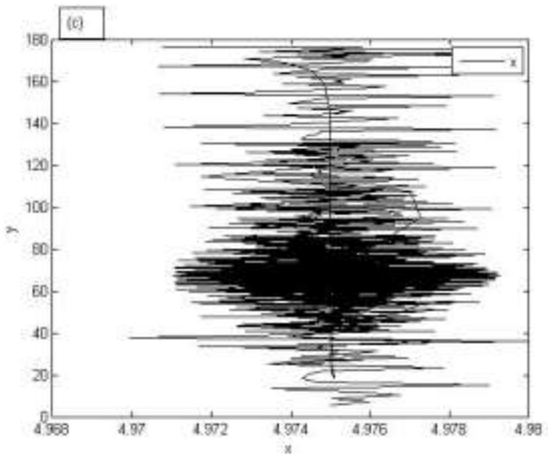

Figure 4(c). 2D xy-plane of figure (5a), stable of the prey and periodic turn to stable of the middle predator.

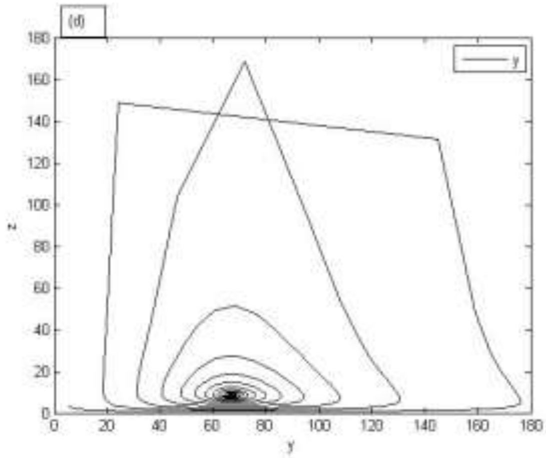

Figure 4(d). 2D yz-plane of figure (5a) periodic change to stable.

5. Results and Conclusions

The model (1) is investigated theoretically and figures of the attractors are blotted in **Figs. 1-3** with data (18) for the modified ratio-dependent with Sokol-Howell and in **Figs. 4(a-d)** with data (20) for modified ratio-dependent with Leslie-Gower. Now, for data (18) we depend on the control parameter the half-saturation level h_4 of the top predator while we depend in data (20) completely, and results after that are obtained:

1) For the value of $h_4 = 1.2$ system (1) shows the periodic as in Fig. 1, while decreasing the value of h_4 from 0.9-05 and 0.4 change the system to less periodic and then to stable as blotted in Fig. 2-3, so the saturation level h_4 is the control parameter of the food chain (1).
2) The permanence of the periodic of the system with fading of the top predator z , so that the model is not complicated as with standard Sokol-Howell functional response.
3) Changing the third term of the model from Sokol-Howell to Leslie-Gower with data (20) turn the system from periodic to stable with coexisting of all the species of the model and less density of the prey x as it plotted in Fig. 4.
4) The model in [8], we used the standard Sokol-Howell with Leslie-Gower and the model exhibits chaotic dynamics while system (1) described above is periodic with nearly the same data.
5) A three of functional responses are used here after putting the Leslie-Gower in the last equation of (1) and Holling type II in the place of Sokol-Howell in the second equation of system (1).

Acknowledgments

The author acknowledged the support from Middle Technical University and Technical Instructor Preparing Institute, Electrical Department.

References

[1] C. Jost, R. Arditi, “Identifying predator-prey process,” Theor. Popul. Biol., vol. 57, no. 4, pp. 325–337, Jun. 1989.

[2] L. JS. Allen, “Introduction to Mathematical Biology,” Pearson-Prentice Hall, New Jersey, 2007.

[3] L. N. Guin, P. K. Mandal, “Spatiotemporal dynamics of reaction-diffusion models of intracting populations,” Appl. Math. Modeling, vol. 38, pp. 4417-4427, 2014

[4] M. Hague, “Ratio-dependent predator-prey models of Intracting populations, ”, Bull. Math. Biol. vol. 7. no.2 pp. 430-452, 2009.

[5] R. . Arditi, R. Ginsburg “Coupling in predator-prey dynamics: ratio-dependence,”. J. Theor. Biol., vol. 139, no. 3, pp. 311–326, Aug. 1989.

[6] S. Gakkhar, R. K. Naji, “Order and chaos in predator to prey ratio-dependent food chain,” Chaos, Soliton & Fractls, vol. 18, pp. 229-239, 2003.

[7] Sokol, W. and Howell, J. A., “The kinentics of phenol oxidation by washed cells,” Biot. Bioe., vol. 30, pp. 921-927, 1987.

[8] S. J. Ali, N. Md. Arifin, R. K. Naji, F. Ismail, N. Bachok, “Analysis of ecological model with Holling type IV functional response,” IJPAM, vol. 106, no. 1, pp. 317-331, 2016.

[9] S. Ruan, D. Xiao, “Global analysis with predator-prey system with non-monotonic functional response, ” SIAM J. Appl. Math. vol. 61, no.4, pp.1445-1472, 2001.

[10] Z. Hu, Z. Teng, L. Zhang, “Stability and bifurcation analysis of discrete predator-prey model with non-monotonic functional response, ” Nonlinear Analysis: Real World Application, vol. 12, pp.2356-2377, 2011.

[11] Z. Zeng, “Dynamics of non-autonomous ratio-dependent food chain, ” J. Appl. Math. & Comput. vol. 215, no.3, pp. 1274-1287, 2009.

[12] C. F. Gerald, PO. Wheatly, Applied numerical analysis, Adison-Wesley, New York, USA, 1998.

[13] R. M. May, Stability and Complexity in model Ecosystems, Princeton University Press, Princeton, New Jersey, 1973.