ON SUBCLASS OF MULTIVALENT HARMONIC FUNCTIONS INVOLVING MULTIPLIER TRANSFORMATION

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Abstract : In this paper , we studied a subclass of multivalent (j-valent) harmonic functions defined by differential operator associated with multiplier transformation , we obtain a coefficients bounds ,distortion bounds and extreme points .
2000Mathematics Subject classification :30C45.
Keywords : Multivalent functions , harmonic function , differential operator,
multiplier transformation.

1. Introduction:

A function $f = u + iv$ is a continuous and a complex valued harmonic function in a complex domain C , if u and so v are real harmonic in C in simply connected domain $R \subset C$, R is domain we can write $f = h + \overline{g}$, where the functions h and g are analytic functions in R . The function h is called analytic part and the function g is called co- analytic part of the function f . A necessary and sufficient condition for f to be locally univalent and sense – preserving in R is that $|h'(z)| > |g'(z)|$ in R . See [6] .Now , we denoted by $RW(j)$ the class of functions defined by the following form: $f = h + \overline{g}$, that are harmonic multivalent and sense – preserving in the unit disk defined as following $U = \{z \in C : |z| < 1\}$. For f belong to $RW(j)$ we may express the functions h and g as following:

$$h(z) = z^j + \sum_{c=j+1}^{\infty} a_c z^c \quad , \quad g(z) = \sum_{c=j+1}^{\infty} b_{ck} z^c \quad , |b_c| < 1. \quad (1)$$

So , for $j \in N, \lambda \ge 0$, the differential operator is defined as following :

$$D_{\lambda}^{n+j-1} f(z) = D_{\lambda}^{n+p-1} h(z) + \overline{D_{\lambda}^{n+j-1} g(z)} \ . \quad (2)$$

When $j = 1$, D_{λ}^{n} denoted of operator introduced by [6]. Also denote $RW^*(j)$ the subclass of $RW(j)$ consisting of all the functions $f = h + \overline{g}$ where h and g defined as :

$$h(z) = z^j - \sum_{c=j+1}^{\infty} |a_c| z^c \quad , \quad g(z) = -\sum_{c=j+1}^{\infty} |b_c| z^c \quad , |b_c| < 1. \quad (3)$$

Now, $D_{\lambda}^{n+j-1} h(z) = z^j + \sum_{c=j+1}^{\infty} [1 + \lambda(c-j)] \varpi(n,c,j) a_c z^c$, (4)

and

$$D_{\lambda}^{n+j-1} g(z) = \sum_{k=j+1}^{\infty} [1 + \lambda(c-j)] \varpi(n,c,j) b_c z^c \ . \quad (5)$$

Where $\varpi(n,c,j) = \binom{c+n-1}{n+j-1}$, $n \in N_0$. (6)

Now , the multiplier transformation $I_j(r,\theta)$ defined as following :

$$I_j(r,\hbar) f(z) = I_j(r,\hbar) h(z) + \overline{I_j(r,\hbar) g(z)} \ . \quad (7)$$

Where

$$I_j(r,\hbar) h(z) = z + \sum_{c=j+1}^{\infty} \Psi(c,j,\hbar)^r a_c z^c , \quad (8)$$

and

$$I_j(r,\hbar) g(z) = z + \sum_{k=j}^{\infty} \Psi(c,j,\hbar)^r b_c z^c , \quad (9)$$

where $\Psi(c,j,\hbar)^r = \left(\dfrac{c+\hbar}{j+\hbar} \right)^r \quad , \hbar \ge 0, r \ge 0$. (10)

So , from (2) and (7) , the Hadmard product defined as following :

$$(D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z) = (D_{\lambda}^{n+j-1} * I_j(r,\hbar)) h(z) + \overline{(D_{\lambda}^{n+j-1} * I_j(r,\hbar)) g(z)}$$
(11)

where

$$(D_{\lambda}^{n+j-1} * I_j(r,\hbar)) h(z) = z + \sum_{k=j+1}^{\infty} \gamma(n,c,j,\hbar)^r a_c z^c \ . \quad (12)$$

And

$$(D_{\lambda}^{n+j-1} * I_j(r,\hbar)) g(z) = z + \sum_{c=j}^{\infty} \gamma(n,c,j,\hbar)^r b_c z^c , \quad (13)$$

where

$$\gamma(n,c,j,\hbar)^r = \varpi(n,c,j) * \Psi(c,j,\hbar)^r \ , \quad (14)$$

Now , we denote by $£_{0,\hbar}^{n,r,æ}(j,◊,¥)$ the class of all functions defined in (1) such that satisfies the following condition :

$$\operatorname{Re}\left\{ \frac{¥\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{j z^{j-1}} + \frac{æ\left[\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1) z^{j-2}\right]}{z^{j-2}} \right\} > ◊ ,$$

(15)

where

$0 < \Diamond < 2¥, ¥ > 0, \lambda \geq 0, \hbar \geq 0, r \geq 0, æ > 0.$

We note that

$£_{0,0}^{0,0,0}(1,0,1) = S_{£}^{*}, H = £$ studied by Silverman [9],

$£_{\lambda,0}^{0,0,0}(1,0,1) = £(\lambda), H = £$ studied by Yalsin and Öztürk [13],

$£_{0,0}^{0,0,0}(1,\Diamond,1) = N_{£}(\Diamond), \Diamond = \alpha$ class studied by Ahuja and Jahangiry [1],

$£_{\lambda,0}^{n,0,0}(1,0,1) = £_{\lambda}^{n}, H = £$ class studied by authors in [7],

$£_{\lambda,0}^{n,0,0}(j,\Diamond,1) = £_{\lambda}^{n}(j,\Diamond), p = j, \Diamond = \alpha, H = £$ class studied by ALshaqsi and Darus in [11].

Also we see that for the analytic part the class $£_{0,\hbar}^{n,r,æ}(j,\Diamond,¥), p = j, \theta = \hbar, \tau = ¥, \mu = æ$ was studied by Goel and Sohi [8].

And so the operator $I_j(r,\hbar)$ was studied by Tehranchin and Kulkarni [12] , Atshan [2], N. E. Cho and T. H. Kim [4], N.E. Cho and Srivastava [5], Saurabh Porwal [10] , J. J. Bhamar and S. M. Khairnar [3].

So ,we denoted by $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ the subclass of $£_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$, where

$$ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥) = RW(j) \cap £_{0,\hbar}^{n,r,æ}(j,\Diamond,¥). \qquad (16)$$

2.Cofficients Bounds:

In the following theorem , we introduced coefficients bounds of a function in the class $£_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Theorem 1:Let $f = h + \overline{g}$, such that the functions h and g are defined in (1) . Let

$$\sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r (|a_c| + |b_c|) \leq j(2¥ - \Diamond) \qquad (17)$$

Where $a_c = \frac{j¥}{¥ + |æ| j(j-1)}$,

$0 < \Diamond < 2¥, ¥ > 0, \lambda \geq 0, \hbar \geq 0, r \geq 0, æ > 0.$

Then f is harmonic multivalent sense preserving in U and f belong to the class $£_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Proof: Let

$$A(z) = \frac{¥\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} + \frac{æ\left[\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}}$$

We using the fact $\operatorname{Re}\{A(z)\} \geq \Diamond$ if and only if $|j - \Diamond + A(z)| \geq |j + \Diamond - A(z)|$.

It suffices to show that

$$|j - \Diamond + A(z)| - |j + \Diamond - A(z)| \geq 0. \qquad (18)$$

So ,

$$\left| j - \Diamond + \frac{¥\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} + \frac{æ\left[\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}} \right| -$$

$$- \left| j + \Diamond - \frac{¥\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} - \frac{æ\left[\left((D_{\lambda}^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}} \right|$$

$$= \left| j + ¥ - \Diamond + \sum_{c=j+1}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r a_c z^{c-j} + \sum_{k=j}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r \overline{b_c z^{c-j}} + \sum_{c=j+1}^{\infty} cæ[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r a_c z^{c-j} + \sum_{c=j}^{\infty} cæ[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r \overline{b_c z^{c-j}} \right|$$

$$- \left| j - ¥ + \Diamond - \sum_{c=j+1}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r a_c z^{c-j} - \sum_{k=j}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r \overline{b_c z^{c-j}} - \sum_{c=j+1}^{\infty} cæ[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r a_c z^{c-j} - \sum_{c=j}^{\infty} cæ[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r \overline{b_c z^{c-j}} \right|$$

$$\geq 2 \left\{ ¥ - \Diamond - \sum_{c=j+1}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |a_c||z|^{c-j} - \sum_{c=j}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |b_c||\overline{z}|^{c-j} - \sum_{c=j+1}^{\infty} |æ| c(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |a_c||z|^{c-j} - \sum_{c=j}^{\infty} |æ| c(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |b_c||\overline{z}|^{c-j} \right\}$$

$$= 2\left\{ j(¥ - \Diamond) - \sum_{c=j+1}^{\infty} c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r |a_c| - \sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r |b_c| \geq 0. \right.$$

So, the harmonic mappings

$$f(z) = z^j + \sum_{c=j+1}^{\infty} \frac{x_c}{c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r} z^c + \sum_{c=j}^{\infty} \frac{\overline{y}_c}{c[1+\lambda(c-j)][¥+|æ| j(c-1)]\gamma(n,c,j,\hbar)^r} (\overline{z})^c. \qquad (19)$$

Where ,

$\sum_{c=j+1}^{\infty} |x_c| + \sum_{c=j}^{\infty} |y_c| = j(¥ - \Diamond)$, show that the coefficient bound given by (17) is sharp.

The function of the form (19) are in $£_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$, because

$$\sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥+|æ|j(c-1)]\gamma(n,c,j,\hbar)^r(|a_c|+|b_c|)$$
$$= p\tau + \sum_{c=j+1}^{\infty} |x_c| + \sum_{c=j}^{\infty} |y_c| = j(2¥-\Diamond).$$

In the next theorem , we show that the condition (17) is also a necessary for functions in the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Theorem 2:Let $f = h + \overline{g}$ where the functions h and g are given by (4). Then a function f belong to the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ if and only if

$$\sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥+|æ|j(c-1)]\gamma(n,c,j,\hbar)^r(|a_c|+|b_c|) \le j(2¥-\Diamond). \quad (20)$$

Where $a_c = \frac{j¥}{¥+|æ|j(j-1)}$,

$0 < \Diamond < 2¥, ¥ > 0, \lambda \ge 0, \hbar \ge 0, r \ge 0, æ > 0.$

Proof : The " if " part follows from theorem 1 , upon noting $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥) \subset £_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$. For the " only if " part , assume that f belong to the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$, then by (15) , we get

$$\mathrm{Re}\left\{ \frac{¥\left((D_\lambda^{n+j-1} * I_j(r,\hbar)) f(z)\right)'}{jz^{j-1}} + \frac{æ\left[\left((D_\lambda^{n+j-1} * I_j(r,\hbar)) f(z)\right)'' - j(j-1)z^{j-2}\right]}{z^{j-2}} \right\} > \Diamond$$

$$\mathrm{Re}\left\{ \tau - \sum_{c=j+1}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |a_c| z^{c-j} - \sum_{c=j}^{\infty} \frac{c¥}{j}[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |b_c| \overline{z}^{c-j} \right.$$
$$\left. - \sum_{c=j+1}^{\infty} æc(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |a_c| z^{c-j} - \sum_{c=j}^{\infty} æc(c-1)[1+\lambda(c-j)]\gamma(n,c,j,\hbar)^r |b_c| \overline{z}^{c-j} \right\} > \alpha$$

We choosing z to be real and so $æ = |æ|$ and letting $z \to 1^-$, we get required result .

In the following theorem , we obtain distortion bounds for the functions in the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Corollary : If $f \in ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$. Then

$$\sum_{c=j}^{\infty} (|a_c|+|b_c|) \le \frac{j(2¥-\Diamond)}{c[1+\lambda(c-j)][¥+|æ|j(c-1)]\gamma(n,c,j,\hbar)^r} \quad (21)$$

Theorem 3: Let f belong the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ and $|z| = r > 1$, then

$$|f(z)| \le (1+a_j)r_1^j + r_1^j \frac{(2¥-\hbar)}{[¥+|æ|j(j-1)]}$$

And

$$|f(z)| \ge (1+a_j)r_1^j - r_1^j \frac{(2¥-\hbar)}{[¥+|æ|j(j-1)]} .$$

Proof:

Let $f \in ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$, so we have

$$|f(z)| \le (1+a_j)r_1^j + r_1^j \sum_{c=j}^{\infty} (|a_c|+|b_c|)$$

Then ,

$$|f(z)| \le (1+a_j)r_1^j + r_1^j \frac{(2¥-\hbar)}{[¥+|æ|j(j-1)]}$$

And so , by similarity we have

$$|f(z)| \ge (1+a_j)r_1^j - r_1^j \frac{(2¥-\hbar)}{[¥+|æ|j(j-1)]}$$

3. Extreme points:

In this section , we shall obtain extreme points for the class $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$.

Theorem 4: $f \in ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ if and only if f can be expressed by

$$f(z) = \sum_{c=j}^{\infty} (S_c h_c + B_c g_c), \quad (22)$$

where

$h_j(z) = z^j$, $h_j(z) = z^j -$
$$- \frac{j(¥-\Diamond)}{c[1+\lambda(c-j)][¥+|æ|j(c-1)]\gamma(n,c,j,\hbar)^r} z^c.$$
$(c = j+1, j+2, \ldots)$

and

$g_k(z) = z^p -$
$$- \frac{j(¥-\Diamond)}{c[1+\lambda(c-j)][¥+|æ|j(c-1)]\gamma(n,c,j,\hbar)^r} (\overline{z})^c.$$
$(c = j+1, j+2, \ldots)$

.

And

$f(z) = \sum_{c=j}^{\infty} (S_c + B_c) = 1$, $S_c \ge 0$, and $B_c \ge 0$, $(c = j+1, j+2, \ldots)$.

In particular , the extreme points of $ð_{0,\hbar}^{n,r,æ}(j,\Diamond,¥)$ are $\{h_c\}$ and $\{g_c\}$.

Proof :

We can write $f(z)$ as following

$$f(z) = \sum_{k=p}^{\infty} (S_c h_c + B_c g_c) = \sum_{c=j+1}^{\infty} (S_c + B_c) z^j -$$
$$- \frac{j(¥-\Diamond)S_c}{c[1+\lambda(c-j)][¥+|æ|j(c-1)]\gamma(n,c,j,\hbar)^r} z^c$$
$$- \sum_{k=p}^{\infty} \frac{j(¥-\Diamond)B_c}{c[1+\lambda(c-j)][¥+|æ|j(c-1)]\gamma(n,c,j,\hbar)^r} (\overline{z})^c$$

$$= z^p - \sum_{c=j+1}^{\infty} \frac{j(¥ - ◊)S_c}{c[1+\lambda(c-j)][¥ + |æ| j(c-1)]\gamma(n,c,j,\hbar)^r} z^c$$

$$- \sum_{c=j}^{\infty} \frac{j(¥ - ◊)B_c}{c[1+\lambda(c-j)][¥ + |æ| j(c-1)]\gamma(n,c,j,\hbar)^r} (\bar{z})^c$$

$$= z^c - \sum_{c=j+1}^{\infty} A_c z^c - \sum_{c=j}^{\infty} C_c (\bar{z})^c \cdot$$

Then from theorem 1 , we have

$$\sum_{c=j+1}^{\infty} c[1+\lambda(c-j)][¥ + |æ| j(c-1)]\gamma(n,c,j,\hbar)^r A_c -$$

$$- \sum_{c=j}^{\infty} c[1+\lambda(c-j)][¥ + |æ| j(c-1)]\gamma(n,c,j,\hbar)^r C_c$$

$$= j(¥ - ◊)\left(\sum_{c=j}^{\infty} (S_c + B_c) - S_c \right)$$

$$= j(¥ - ◊)(1 - S_c) \le j(¥ - ◊) \cdot$$

Then $f \in T_{\lambda,\theta}^{n,r,\mu}(p,\alpha,\tau)$.

Conversely , let f belong to the class $ð_{0,\hbar}^{n,r,æ}(j,◊,¥)$.

Put

$$S_c = \frac{c[1+\lambda(c-j)][¥ + |æ| j(c-1)]\gamma(n,c,j,\hbar)^r}{j(¥ - ◊)} |a_c| ,$$

$(c = j+1, j+2, \ldots)$

And

$$C_c = \frac{c[1+\lambda(c-j)][¥ + |æ| j(c-1)]\gamma(n,c,j,\hbar)^r}{j(¥ - ◊)} |b_c|$$

, $(c = j+1, j+2, \ldots)$ ·

We obtain

$f(z) = \sum_{c=j}^{\infty} (S_c h_c + C_c g_c)$ as required .

So the proof is complete.

References

[1] O. Ahuja and J. Jahangiri, *Noshiro-type harmonic univalent functions,*Sci. Math. Japon. 65(2), (2002), 293-299.

[2] W. G. Atshan and A. S. Joudah, *subclass of meromorphic univalent functions defined by hadamard product with multiplier transformation*, Int. math. Forum, 6(46)(2011), 2279-2292.

[3] J. J. Bhamare and S. M. Khairnar, *Some Properties of Univalent Analytic Functions Defined by Multiplier Transformation*, Sohag J. Math. 5, No. 1, (2018),15-22

[4] N. E. Cho and T. H. Kim, *Multiplier transformation and strongly close - to -convex functions* , Bull Korean Math . Soc. 40 (3)(2003), 399-410.

[5] N.E. Cho and Srivastava , *Argument estimates of certain analytic functions defined by a class of multiplier transformation*, Math.Comput. modeling , 37(1-2)(2003) 39-49.

[6] J. Clunie and T. Shell-Small, *Harmonic univalent functions*, Ann. Acad.Aci. Fenn. Ser. A I Math. 9, (1984), 3-25.

[7] M. Darus and K.Al Shaqsi, On harmonic univalent functions defined by a generalized Ruscheweyh derivatives operator, Lobachevskii Journal of Mathematics 22, (2006), 19-26.

[8] R. Goel and N.Sohi, *New criteria for p-valent*, Indian. J. Pure appl.Math. 7, (2004), 55-61.

[9] H. Silverman, *Harmonic univalent functions with negative coefficients,*Proc. Amer. Math. Soc. 51, (1998), 283-289.

[10] Saurabh Porwal, *On a new subclass of harmonic univalent functions defined by Multiplier transformation*, Mathematica Moravica Vol. 19-2 (2015), 75–87

[11] K. Al Shaqsi and M. Darus, *A New Class of Multivalent Harmonic Functions*, General Mathematics Vol. 14, No. 4 (2006), 37–46.

[12] A. Tehranchi and S..R.Kulkarni, *An application of differential subordination for the class of the meromorphic multivalent functions with complex order* , J. Southeast Asian Bulletin of math. 32(2008), 379-392.

[13] S. Yal¸cin and M. ¨Ozt¨urk, *A new subclass of complex harmonic functions*, Math. Ineq. Appl. 7, (2004), 55-61.