ON ESSENTIAL (T-SMALL) SUBMODULES

Firas sh. Fandi 1, Sahira M. Yaseen 2
1 Mathematics Department College of Pure and Applied Sciences, University of Anbar, Iraq
2 Mathematics Department College of Education for Science, University of Anbar, Iraq
Frissikar798@gmail.com
Sahira.mahmood@gmail.com

Abstract: Let M be an R-module and T be a submodule of M. A submodule K of M is called ET-small submodule of M (denoted by K ≪ETM), if for any submodule K of M (K ≤ M) such that K + N = M implies that N = M.

Keywords: T-small submodule, T-maximal submodule, T-Radical submodule, ET-small submodule, ET-maximal submodule, ET-Radical submodule.

1. Introduction
Throughout this paper R is a commutative ring with identity and M a unitary R-module. A proper submodule N of M is called small (N ≪ M), if for any submodule K of M (K ≤ M) such that K + N = M implies that N = M.

A submodule N of M is essential (K ≤ M) if K ∩ N = 0, then L = , for every L ≤ M [1]. A submodule N of M is closed (N ⊆ M) if N has no proper essential extensions inside M, if is the only solution of the relation N = M. The submodule N of M is called an essential submodule of M such that T = N + T implies T = M [3].

In [4] the authors introduced the concept of small submodule with respect to an arbitrary submodul, that is a submodule K of M is called T-small in M, denoted by K ≪ T M, in case for any submodule X of M, such that T ⊆ K + X, implies that T ∩ X.

In this work we introduce essential T-small (ET-small) submodule, where an R-module M and T be a submodule of M. A submodule K of M is called ET-small submodule of M (denoted by K ≪ET M), if for any essential submodule X of M, such that T ⊆ K + X implies that T ∩ X.

In the first section , we give the fundamental properties of ET-small submodules, Also we give many relations between ET-small submodule and other kinds of small submodules. In the second section, we introduce essential T-maximal (ET-maximal) submodules and the essential T-radical (ET-radical ) submodules of M denoted by RadETM, We give the fundamental properties of this concepts.

2. Essential T-small submodule.

Definition 2.1: Let M be an R-module and let T be a submodule of M. A submodule K of M is called T-small submodule of M (denoted by K ≪ET M), if for any essential submodule X of M such that T ⊆ K + X implies that T ∩ X.

Remarks and Examples 2.2: 1. Consider Z6 as Z-module . Let T= {0, 3}, K= {0, 2, 4}. The only essential submodule of Z6 is Z6 if T ⊆ K + Z6, then T ∩ Z6 = {0, 3}. Thus K ≪ET Z6.
2. It is clear that Every T-small submodule of M is ET-small submodule of M but the converse is not true as for the following Consider Z24 as Z-module and Let T={0, 8, 16 }, N=8Z24, the only essential submodule in Z24 are 2Z24, 4Z24 and 8Z24, T=8Z24 ⊆ 2Z24 and 2Z24 and 8Z24 ⊆ 2Z24, also 8Z24 ⊆ 8Z24 + 4Z24, 8Z24 ⊆ 4Z24 and 8Z24 ⊆ 3Z24. Then 8Z24 ET-small submodule of Z24 which is not T-small submodule of Z24 since 8Z24 ⊆ 8Z24 + 3Z24 but 8Z24 ⊆ 3Z24.

3. Let M be an R-module and T=0, then every essential submodule of M is ET-small in M.
4. Let M be an R-module and T=M, then N ≪ET M if and only if N ≪ M.

Proposition 2.3: Let M be an R-module and let T,H and L be submodules of M such that T ≤ N and H ≤ N ≤ M and N ≪ M, If H ≪ET M, then H ≪ET N.

Proof: Let H be ET-submodules of M and X be an essential submodule of M such that T ⊆ H + X, since X ≤ N and N ≪ M so X ≤ M[2], then H ≪ET M, and T ∩ X.

Proposition 2.4: Let M be an R-module with submodules N ≤ H ≤ M such that T ∩ H . If N ≪ET H, then N ≪ET M.

Proof: Suppose that T ⊆ N + X, for any essential submodule X of M. Since T ∩ H, then T ∩ H contains N + X ∩ H by modular law, since X ≤ N and H ≤ N, then (X ∩ H) ≤ (M ∩ H) = H [2], and N ≪ET H, then T ∩ X. Thus N ≪ET M.

Proposition 2.5: Let M be an R-module and let T, N, and N be submodules of M. Then N ≤ET M and N ≤ET M if and only if N + N ≤ET M.

Proof: Suppose that N ≤ET M and N ≤ET M and Let T ⊆ (N + N) + X, for any essential submodule X of M, then T ∩ (N + N) + X ≤ M, since X ≤ M and N ≤ N, then N + X ≤ M [2] and N ≤ET M. Then T ⊆ N + X, since N ≤ET M. Then T ∩ X. Conversely, let N + N ≤ET M, to show that N ≤ET M and N ≤ET M. Suppose that T ⊆ N + X, for any essential submodule X of M, since N ≤ N + N so T ⊆ N + N + X, but N + N ≤ET M, so T ⊆ X, thus N ≤ET M, and the same we have N ≤ET M.

Proposition 2.6: Let M be an R-module and let H be a submodule of M. If {T} is a family set of submodules of M such that H ≤ET M, for each i ∈ I, then H ≤ET (Σi∈I Ti).

Proof: Let (Σi∈I Ti) = H + X, for any essential submodule X of M. Then for each i ∈ I, Ti ⊆ H + X by hypothesis T ⊆ X, thus (Σi∈I Ti) ⊆ X.

Proposition 2.7: Let M and N be any R-modules and f : M → N be a homomorphism. If T and H are submodules of M such that H ≤ET M, then f(H) ≤ET (f(T)).

Proof: Let f (T) ≠ 0 and f (T) ⊆ (f (H) + X), for any essential submodule X of N, to show T ⊆ H + f (X), let t ∈ T, then t+h+w for some h ∈ H and b ∈ f (X). Hence f (t) = f (h + b) = f (h) + f (b). Thus f (t) = f (h) = f (b), thus f (t) = f (b) ∈ X and so (t-h) ∈ f (X) implies that t ∈ H + f (X)
1(X), since $X \subseteq N$ Thus $f^{-1}(X) \subseteq M$ [2] and $H \ll E_M$.

Therefore $T \subseteq f^{-1}(X).$ Thus $f(T) \leq X$.

**Theorem 2.8:** Let $M$ be an $R$-module and let $T$, $H$ and $N$ be submodules of $M$ such that $H \subseteq N \subseteq M$ and $H \subseteq T$, if $N \ll E_M$ then $H \ll E_M$ and $X \ll E_M$.

**Proof:** Let $N \ll E_M$. To show that $H \ll E_M$, let $T \subseteq X$ for any essential submodule $X$ of $M$. But $H \subseteq N \subseteq M$, so $T \subseteq N \subseteq T$. Thus $H \subseteq T$. Now to show that $X \subseteq N$.

Let $X \subseteq T$, for any essential submodule $X$ of $M$. Since $X \subseteq N$ and $N \ll E_M$, thus $X \subseteq T$.

**Theorem 2.9:** Let $M$ be an $R$-module and let $T$, $H$ and $N$ be submodules of $M$ such that $H \subseteq N \subseteq M$ and $H \subseteq T$ and $H \subseteq M$, if $N \ll E_M$ then $H \ll E_M$.

**Proof:** Let $H \ll E_M$ and $N \ll E_M$.

To show that $N \ll E_M$, let $T \subseteq N$, for any essential submodule $X$ of $M$ and $H \subseteq X$. Now $T \subseteq X$, since $X \subseteq M$ and $H \subseteq M$, thus $X \ll E_M$.

Let $X \ll E_M$.

**Proposition 2.10:** Let $M$ be an $R$-module and let $K$ and $H$ be submodules of $M$ such that $K \ll E_M$ and $H \ll E_K$. Then $(H \cap K) \ll E_{(H \cap K)}$.

**Proof:** Let $K \ll E_M$ and $H \ll E_M$, since $(H \cap K) \subseteq H$ and $(H \cap K) \subseteq K$, by Proposition 1.4. $(H \cap K) \subseteq E_M$. Also by Proposition 1.6 we get $(H \cap K) \subseteq E_{(H \cap K)}$.

**Proposition 2.11:** Let $M$ be an $R$-module and let $T$, $K$, $H$ and $B$ be submodules of $M$ such that $K \subseteq H \subseteq B \subseteq M$, $K \subseteq M$ and $H \subseteq M$. Then $B \subseteq E_{(T \cap K)}$.

**Proof:** Let $B \subseteq E_{(T \cap K)}$. To show that $B \subseteq E_{(T \cap K)}$, let $T \subseteq B$. Since $B \subseteq E_{(T \cap K)}$, then $B \subseteq E_{(T \cap K)}$.

Therefore $B \subseteq B + X$, since $X \subseteq M$ and $K \subseteq M$ then $B \subseteq E_{(T \cap K)}$.

**Theorem 2.12:** Let $M = M_1 \oplus M_2$ be a $R$-module such that $R = Ann(M_1) + Ann(M_2)$. If $N \ll E_{(M_1)} \oplus N \ll E_{(M_2)}$, then $N \ll E_{(M_1 \oplus M_2)}$.

**Proof:** Let $T \subseteq T_1 \subseteq N_1 \oplus N_2 \subseteq X$, for any essential submodule $X$ of $M$. Since $R = Ann(M_1) + Ann(M_2)$, then, by the same argument of the proof of [6, prop. 4.2, ch 1] $X = X_1 \oplus X_2$, for any essential submodule $X_1$ of $M_1$ and submodule $X_2$ of $M_2$. Hence $T_1 \subseteq T_2 \subseteq (N_1 + X_1) \oplus (N_2 + X_2)$ to show that $T \subseteq E(T_1 \oplus N_1 \oplus N_2 \oplus X)$, $T \subseteq T_2 \subseteq (N_1 + X_1) \oplus (N_2 + X_2)$, since $t_1 \subseteq (N_1 + X_1)$ and $t_2 \subseteq (N_2 + X_2)$, hence $T \subseteq (N_1 + X_1)$ and $T \subseteq (N_2 + X_2)$, hence $T \subseteq E(T \subseteq K(X) \subseteq X$. Thus $N \ll E_{(M_1 \oplus M_2)}$.

**Proposition 2.13:** Let $M = E_{(M_1, M_2)}$ be a fully stable module if for each submodule $K$ of $M$ and each $R$-homomorphism $f$ from $M$ into $K$, $f(K) \subseteq K$ [5].

**Proof:** Let $M = E_{(M_1, M_2)}$ be a fully stable module and $K \subseteq E_{(M_1)} \subseteq M_1$. For each $iE_1$, then $E_{(M_1, M_2)} \subseteq E_{(M_1, M_2)}$. Let $E_{(M_1, M_2)} \subseteq E_{(M_1, M_2)}$, for any essential submodule $X$ of $M$. Claim that $X \subseteq E_{(M_1, M_2)}$. To show that, for each $iE_1$ let $P_i : M \rightarrow M_1$ be the projection map and let $x \in X$ and hence $x = \sum_{i \in I} r_i x_i$ where $x_i \in M_i$, and $r_i \in I$ for at most a finite number of $I$. Since $M$ is fully stable, then $P_i(x) = x$, for all $i$. Now $P_i(x) = P_i(\sum_{i \in I} x_i) = x_i \in X \subseteq M_1$ and hence $x = \sum_{i \in I} r_i x_i \in E_{(M_1, M_2)}$. Clearly $\subseteq E_{(M_1, M_2)}$. Therefore $X \subseteq E_{(M_1, M_2)}$.

Recall that the annihilator of $M$ $Ann(M) = \{ r \in R \mid rM = 0 \}$, $M$ is a faithful module if $Ann(M) = 0$.

**Proposition 2.14:** Let $M$ be a finitely generated, faithful and multiplication module and let $I$ be ideals in $R$. Then $I \ll E_{(M)}$ if and only if $IM \subseteq E_{(M)}$.

**Proof:** Let $I \subseteq E_{(M)}$. To show that $IM \subseteq E_{(M)}$, let $JM \subseteq X$, for any essential submodule $X$ of $M$. Since $M$ is multiplication module, then $X = KM$, for some ideal $K$ and hence $JM \subseteq M + KM = (1 + K)M$. Since $M$ be a finitely generated, faithful and multiplication module, therefore $M$ is a cancellation module, by $[9]$, then $J \subseteq K$ since $K \subseteq R$ since $I \subseteq M$. Then $JM = X$, Thus $IM \subseteq E_{(M)}$. Conversely, let $IM \subseteq E_{(M)}$. To show that $I \subseteq E_{(M)}$. Let $K$ be the essential ideal of $R$ such that $J \subseteq K$. Since $M$ is multiplication module, then
3. Essential T-Radical of M. Recall that if M is an R-module and T be a submodule of M. A submodule K of M is called T-maximal submodule of M if T ⊈ K . Thus T ⊈ K .

4. Consider Z₆ as Z-module . Let T= Z₆ and K= { 0, 2 } , then K is the only ET-maximal submodule of Z₆. To show that, Z₆+ K= Z₆ and Z₆+ K= Z₆ .

5. Consider Z₅ as Z-module . Let T= Z₅∞ , then Z₅∞ has no ET-maximal submodule and hence Rad₅(Z₅∞) ⊈ Z₅∞.

Proposition 3.10: Let M be an R-module and let T be a finitely generated submodule of a module M. Then \( \text{Rad}_{\text{ET}}(M) \subseteq \text{ET} M \).

Proof: Assume that T \subseteq \text{Rad}_{\text{ET}}(M)+X. For any essential submodule X of M, to show that T \subseteq X suppose that T \nsubseteq X. Then by Proposition (2.7), there exists a ET-maximal submodule K of M such that X \subseteq K. Therefore T \subseteq \text{Rad}(M)+X \subseteq K implies that T \subseteq K, so \( \frac{R+K}{K} = 0 \) which contradicts the T-maximality of K. Thus T \subseteq X. Therefore \( \text{Rad}_{\text{ET}}(M) \subseteq \text{ET} M \).

Lemma 3.11: Let M be an R-module and let T be a submodule of M. Assume that T \subseteq \text{Rad}_{\text{ET}}(M)+X, then there exists H is a ET maximal (ET-maximal) submodule of M containing H.

\[ \text{Rad}_{\text{ET}}(M) \subseteq \text{ET} M \]

Proof: Assume that T \subseteq \text{Rad}_{\text{ET}}(M)+X. For any essential submodule X of M, to show that T \nsubseteq X suppose that T \nsubseteq X. Then by Proposition (2.7), there exists an ET-maximal submodule H of M such that X \subseteq H. Therefore T \subseteq \text{Rad}(M)+X \subseteq H implies that T \subseteq H, so \( \frac{R+H}{H} = 0 \) which contradicts the T-maximality of H. Thus T \subseteq H. Therefore \( \text{Rad}_{\text{ET}}(M) \subseteq \text{ET} M \).

Proposition 3.12: Let M and N be R-modules and f: M \rightarrow N be an R-epimorphism such that Kerf \subseteq \text{Rad}_{\text{ET}}(M). Then f(\text{Rad}_{\text{ET}}(M)) = \text{Rad}_{\text{ET}}(N).

Proof: Since f is epimorphism, by Proposition (2.5) and Proposition(2.6), we have f(\text{Rad}_{\text{ET}}(M)) = \text{Ef}(\text{Rad}_{\text{ET}}(M)) = \text{Ef}(f(\text{Rad}_{\text{ET}}(M))) = \text{Ef}(\text{Rad}_{\text{ET}}(N)) = \text{Rad}_{\text{ET}}(N), where A = \{ K \leq M | K is an ET-maximal submodule of M \} and B = \{ f(K) \leq N | f(K) is an Ef(T)-maximal submodule of N \}.

References


