The Unique Maximal J-Regular Submodule

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Abstract: An R-module A is said to be J-regular module if, for each $a \in J(A)$, $r \in R$, there exist $t \in R$ such that $r\,a = r\,t\,r\,a$. We proved that each unitary R-module A contains a unique maximal J-regular submodule, which we denoted by M (A). Furthermore, the radical properties of A have investigated. We proved that if A is an R-module and N is a submodule of A, then J (N) ∩ M (A) ⊆ M (N). Moreover, if A is "projective," then M (A) = M (R) · J (A) and M (A) ∩ J(R).J (A) = (0).

Key Words: pure submodules**,** J-pure submodules, regular modules, J-regular modules.

Introduction

Throughout this paper, R is a commutative ring with identity and all modules are left, unitary, unless otherwise stated. An element $r \in$ R is said to be regular if there exists $t \in$ R such that $rtr = r$; a ring R is called regular if and only if each element of R is regular. An ideal I of a ring R is regular if each of its elements is regular in R; indeed, a regular ideal I of R is itself a regular ring [1]."Brown and McCoy proved in" [1] that each ring R contains a unique maximal regular ideal M(R), which satisfies the well-known radical properties. The ideal M(R) is called the regular radical of R. The concept of regularity extended to modules in several ways and in [2] the notion of F-regular modules (in the sense of Fieldhouse [3]) generalized to GF-regular modules . Let M be an R-module; an element $x \in$ M is, said to be GF-regular if for each $r \in$ R there exist $t \in$ R and a positive integer n such that $r^n t r^n x = r^n x$. An R-module M is called GF-regular if and only if all its elements are GF-regular. In [2] that each module contains a "unique maximal GF-regular submodule".

An R-module M is said to be J-regular module if for each $m \in J(M)$, $r \in R$, there exist $t \in R$ such that r m = r t r m [4].

A submodule N of an R- module A is called J- regular if each element of N is J- regular and every submodule of a J- regular module is a J- regular module. Also, the concept of J- pure submodule has been introduced. A submodule N of an R-module M is called a J-pure if N is pure in J (M), i.e. for each ideal I of R, I J (M) ∩ N = IN, where J (M) is the Jacobson radical of M [4]. In this paper, we show that each module contains a "unique maximal J- regular submodule," which we denoted by M (A), and we show that M (A) satisfies some but not all of the usual radical properties .

1. Main Results

Definition 1.1. Let A be an R -module. The unique maximal J-regular submodule of a module A denoted by M (A). If there exist a submodule containing every J-regular submodule of A, this means that M (A) is a J-"regular submodule" which is not contained properly in any J-"regular submodule".

Remarks and Examples 1.2

(1) If A = R, then M (A) is an ideal of R .

(2) It is clear that A is J- regular R- module if and only if M (A) = A.

(3) Since the Z-module Z_4 is J-"regular" [4]. Then M (Z_4) = Z_4.

(4) Each "submodule in the" Z -module Q is not J-regular, hence M (Q) = (0). Suppose that, M (Q) = B for some submodule B of Q implies that B is J-"regular" as Z-module. Take any element $x \in$ J (B), $x = \frac{a}{b}$ where a and b are two non – zero elements in Z. If we take an ideal <n> of Z where n is greater than one, then the non – zero cyclic submodule generated by $\frac{a}{b}$ is not J- pure in B, that is $< n >$ J (B) $\cap <\frac{a}{b}> \neq\ < n ><\frac{a}{b}>$ which is a contradiction as B is J-regular.

(5) The Z– module Z is J- regular since J (Z) = 0 [4], then by remark 1.2 M (Z) = Z.

(6) The module Z_{P^∞} as Z –module is not J- regular. To show that, let $G_n = <\frac{1}{P^n} + Z>$ be any submodule of Z_{P^∞} where P is a fixed prime number and n is a positive integer . Then $\frac{1}{P^n} + Z = P^n\left(\frac{1}{P^{2n}} + Z\right) \in P^n J(Z_{P^\infty}) \cap G_n$, but $\frac{1}{P^n} Z \notin P^n < \frac{1}{P^n} + Z > Z\ = 0_{Z_{P^\infty}}$.

Thus M (Z_{P^∞}) $\neq Z_{P^\infty}$. Now, since every submodule $G_n = <\frac{1}{P^n} + Z>$ of Z_{P^∞} is isomorphic to the module Z_{P^∞} as Z –module, where $Z_P \subset Z_{P^2} \subset Z_{P^3} \subset \cdots \subset Z_{P^n} \subset \cdots$ .Thus M (Z_{P^∞}) $\cong Z_{P^n}$ for some integer n ≥ 0, this follows from the fact that $Z_{P^\infty} = \bigcup_{n \geq 0} Z_{P^n} = \sum_{n\geq 0} < \frac{1}{P^n} + Z >$.

(7) If B is a J- regular submodule of an R-module A, then B is not necessary be a J- pure submodule of A. For example, consider the module Z_8 as Z– module, let B = $< \bar{4} > = \{\bar{0}, \bar{4}\} \cong Z_2$ be J- regular module, but B is not J- pure submodule of Z_8, since if I= 2Z is an ideal of Z, then $IJ(Z_8) \cap \{\bar{0}, \bar{4}\} = 2\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \cap \{\bar{0}, \bar{4}\} \not\subseteq \{\bar{0}\}$. I $\{\bar{0},\bar{4}\} = 2\{\bar{0},\bar{4}\} = \{\bar{0}\}$. Hence, the maximal J- regular submodule of Z_8 is M (Z_8) = $< \bar{2} > + < \bar{4} > \ = < \bar{2} >$.

(8) Let A be an R-module and $M^*(A)$ be the maximal regular submodule of A then it may be $M^*(A) \neq M(A)$. For example the module Z_8 as Z-module. It is easily to show that the regular submodules of Z_8 are $< \bar{0} >$ and $< \bar{4} >$ since $< \bar{4} > \cong Z_2$, thus $M^*(Z_8) = < \bar{4} >$. But $M(Z_8) = < \bar{2} >$, therefor, $M^*(Z_8) \neq M(Z_8)$.

Theorem 1.3. Every R-module contains a unique maximal J-regular submodule.

Proof: Let A be an R-module and $G = \{N: N$ is a J-regular submodule of $A\}$. Notice that as (0) is a J-regular submodule then G is a non-empty set. Let $\{Ni\}$ be an ascending chain in G and $B = \bigcup_{i\in\Lambda}$ Ni. Let $b \in$ J (B) then $b \in$ J ($\bigcup_{i\in\Lambda}$ Ni) = $\bigcup_{i\in\Lambda}$ J(Ni). In particular, if Ni , for each i = 1,2. To show J($N_1 \cup N_2$)=J(N_1)∪J(N_2). Let $x \in J(N_1 \cup N_2)$ then R_x is small in $N_1 \cup N_2$ by [3].There exist a submodule K of $N_1 \cup N_2$, such that $N_2 = N_1 \cup N_2 = R_x + K$, then implies $x \in J(N_2)$. Hence $J(N_1 \cup N_2) \subseteq J(N_1) \cup J(N_2)$.

Conversely, assume that $y \in J(N_1) \cup J(N_2)$, then either $y \in J(N_1)$ or $y \in J(N_2)$. If $y \in J(N_2)$, then R_y a small in $N_2 = N_1 \cup N_2$. So we obtain is a small in $N_1 \cup N_2$. Hence $y \in J(N_1 \cup N_2)$. There exists $j \in \Lambda$ such that $b \in J(N_j)$, but N_j is a J- regular submodule, then for each $r \in R$, there exist $t \in R$ such that $rtrb = rb$ therefore b is a J- regular element in B which implies that B is a J-regular R-module. Now, by Zorn's Lemma, G contains a maximal element, which we call it $\mathcal{M}$. To prove the uniqueness of $\mathcal{M}$, assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ be two "maximal J-regular submodules" in A, then for any maximal ideal P of R each of $\mathcal{M}_1p$ and $\mathcal{M}_2p$ "is semisimple over" R_p [4]. Now, let $\mathcal{M}_1p \cap \mathcal{M}_2p = K_p$; then $K_p \subseteq \mathcal{M}_1p$ and $K_p \subseteq \mathcal{M}_2p$, thus $\mathcal{M}_1p = K_p + A_1p$ and $\mathcal{M}_2p = K_p + A_2p$, where A_1p and A_2p are two submodules of Ap [5]. Hence, $\mathcal{M}_1p + \mathcal{M}_2p = A_1p + K_p + A_2p$, but each of A_1p, A_2p, and K_p is a"semisimple submodule;" thus $\mathcal{M}_1p + \mathcal{M}_2p$ is a "semisimple submodule" which implies that $\mathcal{M}_1p + \mathcal{M}_2p$ is J-"regular" [5]. So $\mathcal{M}_1 + \mathcal{M}_2$ is a J-"regular submodule" [4]. Now, as of $\mathcal{M}_1$ and $\mathcal{M}_2$ "is a maximal J-regular submodule" and hence $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{M}_2 = \mathcal{M}_1$.

Proposition 1.4. Let A be an R-module and N a submodule of A, then J (N) ∩ M $(A) \subseteq$ M (N).

Proof: Let $x \in J(N)$ and $x \in M(A)$, thus for each $r \in R$, $rx = rtrx$ for some $t \in R$. Then x is J- regular element in N, which means that $x \in$ M (N).

Proposition 1.5. Let A_1 and A_2 be R-modules, then M $(A_1 \oplus A_2) \subseteq$ M $(A_1) \oplus$ M (A_2).

Proof: Let $c \in$ M $(A_1 \oplus A_2)$ and A = $A_1 \oplus A_2$,then $c = (a, b)$, where $a \in A_1$ and b $\in A_2$. Since c is J-regular element in A, then each of a and b is J- regular element, in A_1 and A_2 , respectively. Which means that $a \in$ M (A_1) and $b \in$ M (A_2), hence $c \in$ M $(A_1) \oplus$M (A_2).

Recall that "the annihilator of an element x of an R-module A denoted by ann(x) is defined to be ann(x) = {r ∈ R: rx = 0} and the annihilator of A denoted by ann(A) is defined to be ann(A) = {r ∈ R: rx = 0 for every $x \in$ A}. Clearly, ann(x) and ann(A) are ideals of R, [6]." In [4] we prove that $\frac{R}{\text{ann}(x)}$ is the regular ring for each $x \in$ J (A) if and only if A is J-regular R-module. In fact if $\frac{R}{\text{ann}(J(A))}$ is a regular ring, then A is J- regular.

Proposition 1.6. Let A and A′ be R- modules, and $f : A \rightarrow$ A' be an R-homomorphism; then f(M(A)) ⊆ M(f(A)).

Proof: Let $a \in$ M (A), then a is J- regular element in A and $a \in$ J (A), which implies that $\frac{R}{\text{ann}(a)}$ is regular ring, for each $a \in$ J (A) [4]. But ann$(a) \subseteq$ ann((a)) and $f(a) \in f$(J (A)) ⊆ J (A'), hence exists φ: $\frac{R}{\text{ann}(a)} \rightarrow \frac{R}{\text{ann}(f(a))}$ define by φ (r +ann(a)) = r + ann(f(a). Since $\frac{R}{\text{ann}(a)}$ is regular ring, for each a ∈ J (A), then by [7] $\frac{R}{\text{ann}(f(a))}$ is regular ring, hence f(A) is J- regular R-module [4] and f(a) ∈ M(f(A)). Thus f(M (A)) ⊆ M ((A)).

Proposition 1.7. Let A be a J- regular R-module, then M $(\frac{A}{M(A)})$ = (0).

Proof: Since A is a J- regular, then M (A) =A. Thus M $(\frac{A}{M(A)})$ = M $(\frac{A}{A})$ = (0).

Remark 1.8. For any R- module A, M $(\frac{A}{M(A)}) \neq (0)$ in general. For examples, the module Z_8 as Z-module M $(\frac{Z_8}{M(Z_8)})$ = M $(\frac{Z_8}{<\bar{2}>}) \cong M(Z_2) = Z_2$. Thus $(\frac{Z_8}{M(Z_8)}) \neq$ (0).

Proposition 1.9. For each R-module A, M (R) · $A \subseteq$ M (A).

Proof: For each $a \in A$, let f_a: R → A be an R-homomorphism defined by $f_a(r) = ra$ for each $r \in R$, then by Proposition 1.6, f_a (M(R)) ⊆ M (A). On the other hand, M (R). A = $\sum f_a$(M(R)). Hence M (R) · $A \subseteq$ M (A).

Remark 1.10. The reverse inclusion M $(A) \subseteq$ M (R) · A, in Proposition 1.9 is not true in general. For example, the module Z_4 as Z-module where M $(Z_4) = Z_4 \not\subseteq$ M (Z) $Z_4 = Z(Z_4)$.

Let J (R) be the Jacobson radical of a ring R. Brown and McCoy proved [1] that M (R) ∩ J (R) = (0), where R is F- regular ring and for R is J-regular ring [4] M (R) ∩ J (R) = (0). However, this is not true for J- regular modules for example, if A = Z_4 as Z-module, then M (A) = Z_4 and J (A) = $\{\bar{0}, \bar{2}\}$ but M (A) ∩ J (A) = $\{\bar{0}, \bar{2}\} \neq (0)$.

Lemma 1.11. Let A be an R-module and N be a J- pure submodule of A. Let I be an ideal of R, then N=IN if and only if N ⊆ IJ(A).

Proof: Since N is J- pure submodule in A, then N ∩ IJ(A) = IN [4], for some ideal I of R. If N = IN, then N ∩ IJ(A) ⊆ N and hence N ⊆ IJ (A).

Conversely, if N ⊆ IJ (A), Then N ∩ IJ(A) = N, but N ∩ IJ(A) = IN; since N is J-pure submodule. Hence N = IN.

Recall that "(Nakayama`s Lemma) for an ideal I of R then I ⊆ J(R) if and only if for every finitely generated R- module M, if IM = M, implies M = $< 0 >$ [5].

Lemma 1.12. Let I be an ideal of a ring R contained in J(R) and let N be a finitely generated J- pure submodule of an R- module A with N ⊆ I J(A) implies that N = $< 0 >$.

Proof: By Lemma 1.11, we obtain N = IN and since I ⊆ J(R) we have N = $< 0 >$ by "Nakayama`s Lemma." If A is "F-regular R-module" and M (A) is "pure submodule" of A, then $M(A) \cap J(R).A = (0)$ [7]. For J- regular R-module we have the following:

Proposition 1.13. Let A be an R-module. If M (A) is a J- pure submodule of A, then M (A) ∩ J (R) ·J (A) = (0).

Proof: Let $x \in$ M (A) ∩ J (R) · J (A), then $x \in$ M (A) and $x \in$ J(R) · J (A). Let N be the cyclic submodule generated by x. It's clear that N ⊆ M (A), since M (A) is J- regular module; then N is J- pure in M (A). But M (A) is J- pure in A so N is J-pure in A [4]. On the other hand, N ⊆ J(R).J(A), hence by Lemma 1.12, we have N = 0. Which implies that M (A) ∩ J(R) · J (A) = (0).

Recall that "an R-module P is said to be a projective module if for any homomorphism f : P → B and for any epimorphism g: A → B; where A and B are two R-modules there exist a homomorphism h: P →A such that $f = g \circ$ h. [8]".

Recall that "(Dual Basis Lemma) an R- module A is projective module if and only if there exists a family of elements { x_i: i∈ Λ} ⊆ M and {f_i: i∈ Λ} ⊂ M^*= Hom (M;R) such that for any $x \in$ M, $f_i(x) = 0$ for almost all i, (equivalently, $f_i(x) \neq 0$ only for a finite number of i∈ Λ and $x = \sum_{i\in\Lambda} x_i f_i(x)$, [8]".

Theorem 1.14. Let A be a projective R-module then

(1) M (A) = M (R) · J (A),

(2) M(A) is a J- pure submodule of A for each ideal I in R.

Proof: (1) Let $x \in$ M (A). $x \in$ J (A). Since A is projective R- module, then by "Dual Basis Lemma" there exists a family { x_i: i∈ Λ} of A and {f_i: i∈ Λ} ⊆ M*= Hom (A, R) where $f_i(x) \neq 0$ only for finite number of i∈ Λ and $x = \sum_{i\in\Lambda} x_i f_i(x)$. But $f_i(x) \in$ M (R) by Proposition 1.6. Thus M (A) ⊆ M (R) ·J (A). We get the other direction of the inclusion by Proposition 1.9. Therefor M (A) = M (R) · J (A).

(2) Let I be an ideal of a ring R.

$\mathrm{M(A)} \cap \mathrm{IJ(A)} = \mathrm{M(R)J(A)} \cap \mathrm{IJ(A)} = (\mathrm{M(R)} \cap \mathrm{I})\mathrm{J(A)}$.

But M(R) is J- pure ideal in R, then $\mathrm{M(R)} \cap \mathrm{I} = \mathrm{IM(R)}$.

Hence $\mathrm{M(A)} \cap \mathrm{IJ(A)} = \mathrm{IM(R)J(A)} = \mathrm{IM(A)}$.

Recall that "if A is an R-module, then the trace of A is tr$(A) = \sum_{f\in A^*} f(A)$, where A*=Hom($A$,$R$),[9]"

Proposition 1.15. Let A be a J- regular R-module. If tr (A) = R, then R is a J-regular.

Proof: Since A is J-regular, then M (A) = A, Remark (1.2) (2) and then M (A) = f (M (A)) ⊆ M(R) by Proposition (1.6) where $f \in A^*$ = Hom(A, R). Thus R = tr (A) = $\sum_{f\in A^*} f$(A) ⊆ M (R) implies R = M (R). Therefor R is a J-regular .

Proposition 1.16. Let A be a finitely generated R-module and M (A) + J (A) = A, then A is J- regular .

Proof: Since A is finitely generated, then J (A) is small submodule of A, but M (A) +J (A) =A, therefore M (A) = A and hence A is J- regular.

Remark 1.17. For any R-module A, M (A) + J (A) ≠ A in general. For example, the module Z_8 as Z- module is not J-regular where M (Z_8) + J (Z_8) = $<\bar{2}> + <\bar{2}> = <\bar{2}> \neq Z_8$.

Recall that "a submodule N of an R- module A is called an essential submodule of A if for each submodule L of A with N∩ L =0 implies L= 0," [8]. We have the following:

Proposition 1.18. Let N be a submodule of an R- module A and J (N) be an essential submodule of A. If M (N) = 0, then M (A) = 0 .

Proof: Since J (N) ∩ M (A) ⊆ M (N) by Proposition 1.4. Then 0 = J (N) ∩ M (A). But J (N) is an essential submodule of A, "thus M (A) = 0.

Recall that a submodule K of an R-module A is said to be stable if $f(K) \subseteq K$ for each R-homomorphism,$f: K \longrightarrow A$" [10]. **Proposition 1.19.** For any R-module A, then M (A) is "stable submodule" of A.

Proof: Let $f \in Hom(\mathrm{M(A)}, \mathrm{A})$. By proposition 1.6, $f(\mathrm{M(M(A))}) \subseteq \mathrm{M(A)}$. But M (M (A)) = M (A) since M (A) is J- regular. Thus, $f(\mathrm{M(A)}) = f(\mathrm{M(M(A))}) \subseteq \mathrm{M(A)}$. Hence M (A) is "stable submodule."

Recall that "a non-zero submodule K of an R-module A is said to be dense in A if K generates A, that is $\mathrm{A} = \sum_{f\in Hom(\mathrm{K,A})} f(\mathrm{K})$" [11].

Proposition 1.20. Let A be an R-module and M (A) be a "dense submodule" in A, then A is J- regular module.

Proof: Since M (A) is "dense" in A, then $\mathrm{A} = \sum_{f\in Hom(\mathrm{M(A),A})} f(\mathrm{M(A)})$. But M (A) is "stable submodule" of A by the previous Proposition 1.19, thus $f(\mathrm{M(A)}) \subseteq \mathrm{M(A)}$

implies $\mathrm{A} = \sum_{f\in Hom(\mathrm{M(A),A})} f(\mathrm{M(A)}) \subseteq \mathrm{M(A)}$. Then $\mathrm{A} = \mathrm{M(A)}$ therefore A is J- regular .

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